Moment Solutions for the State Exiting Counting Processes of a Markov Renewal Process

Manuel D. Rossetti (mr2m@virginia.edu)
Department of Systems Engineering, Thornton Hall, University of Virginia, Charlottesville, VA 22903

Gordon M. Clark
Department of Industrial, Welding, and Systems Engineering, The Ohio State University, 210 Baker Systems, 1971 Neil Avenue, Columbus, OH 43210

Abstract. Important performance measures for many Markov renewal processes are the counts of the exits from each state. We present solutions for the conditional first, second, and covariance moments of the state exiting counting processes for a Markov renewal process, and solutions for the unconditional equilibrium versions of the moments. We demonstrate the relationship between the conditional first moments for the state exiting and the state entering counting processes. For analytical and illustrative purposes, we concentrate on the two state case. Two asymptotic expansions for the moment functions are proposed and evaluated both analytically and empirically. The two approximations are shown to be competitive in terms of absolute relative error, but the second approximation has a simpler analytical form which is useful in analyzing more complex stochastic processes having an underlying MRP structure.

Keywords: Markov Renewal Processes, Moment Approximation, Counting Processes

1. Introduction

Markov renewal theory combines the theory of renewal processes and Markov chains and has a considerable literature history; see for example Pyke (1961a), (1961b) and also Çinlar (1969), (1975a), (1975b) for the early theoretical development and other references. Markov renewal processes (MRP) have been applied in the study of the reliability of systems, the study of population movement, and in the study of queueing systems as well as in many other areas. For a bibliographical reference of applications, see Cheong (1974). For more recent work and references, we refer the reader to Narayana and Neuts (1992). Our interest in MRP's is derived from our investigations into the estimation of queueing system service parameters from queue departure processes; see Rossetti and Clark (1995) for more details.

This paper considers the development of Laplace-Stieltjes transforms for the moments of the number of times a given state in a special Markov renewal process is exited. We also develop two approximations based on asymptotic expansions for the moments. The approximations are analyzed both analytically and empirically in order to evaluate their accuracies. The approximations can be used by practitioners when making decisions based upon the state exiting counting processes of a Markov renewal process, and also by researchers when analyzing other stochastic models utilizing an underlying Markov renewal process structure, see e. g. Rossetti and Clark (1995).

The rest of this paper is structured as follows. First, we give the contextual problem which motivated our research. In Section 2, we present the definition and moment solutions for the state exiting counting process associated with a Markov renewal process. In Section 3, we present asymptotic expansions for the moments of the two state Markov renewal process and exact analytical expressions for the case in which the sojourn time distribution function is negative exponential. In Section 4, we present and evaluate approximations for the moments both analytically and empirically. Section 5 illustrates the use of the results via two applications. Finally, we summarize our results.

2. Definitions and Solutions

We begin this section with an example which motivates the definition of the MRP examined in this paper. We then define the state exiting counting process (ECP) associated with the MRP and present solutions for the moments of the counting processes.

2.1. Motivating Example

This example is motivated by research involving the estimation of the mean service time in a multi-class customer queueing system and in the use of the departure process for planning and control. Assume that we have a server which processes heterogeneous customers, and that the selection of the next customer occurs according to a Markov chain. After selecting the customer, the server spends a random amount of time servicing the customer where the service time distribution depends only upon the type of customer selected. If we define the state of the process as the type of customer the server is currently serving, then the sojourn time distributions for the states of the MRP correspond to the service time distribution functions for the customers. As an additional constraint, let us assume that a system which monitors the server exists but is only able to record the count of the number of each type of customer which completes service during a fixed interval of time. Such a constraint is quite prevalent in production systems which have shop floor inventory tracking and control systems. In essence, these systems can record the total production of each type of customer during time intervals such as a shift but due to cost or labor practices it becomes infeasible to record counts during smaller time intervals. A fundamental decision in the operation of a production system is the scheduling of the production of the various types of customers without exceeding the capacity of the system. Knowledge about the relationships between the production counting processes of the customers can be extremely useful in the decision making process. The production process of each type of customer is really the count of the number of times each state is exited. In the next section, we give a precise definition for the exiting counting processes described in this example.

2.2. Exiting Counting Process Definition

We begin with some definitions and notation for the state entering counting process associated with the MRP and then precisely define the counting process for the number of times a state is exited for the MRP examined in this paper.

Adapting the notation used in Çinlar (1975b), (1968), we give the following definition for the state entering counting process.

DEFINITION 1. *Let* $\{X_n, n \geq 0\}$ *be a Markov chain with finite state space,* $S = \{1, 2, \ldots, K\}$*. Let* $\{Y_n, n \geq 0\}$ *be a sequence of non-negative random variables where* $n \geq 1$ *and* $Y_0 = 0$*. Let* $T_n = Y_0 + Y_1 + \cdots + Y_n$, $n \geq 0$*. Let* $(X, T) = \{X_n, T_n; n \geq 0\}$ *be a Markov renewal process with the property*

$$\begin{aligned} & \Pr\{T_{n+1} - T_n \leq t \mid X_0, \cdots, X_n; T_0, \cdots, T_n\} \\ = \; & \Pr\{T_{n+1} - T_n \leq t \mid X_n\} \end{aligned} \tag{1}$$

Let $\{X(t), t \geq 0\}$ *be the semi-Markov process engendered by* (X, T)*, where* $\{X(t), t \geq 0\}$*, is defined as* $X(t) = X_n$ *if and only if* $T_n \leq t \leq T_{n+1}$*. The random variables* T_n *and* Y_n *are called the* n^{th} *transition epoch and the* n^{th} *sojourn time of the process and* $F_i(t) = \Pr\{Y_n \leq t \mid X_{n-1} = i\}$ *is called the sojourn time distribution function for state* i*. Define* θ_i *and* δ_i^2 *to be the mean and the variance of the* n^{th} *sojourn time,* Y_n*, whose distribution function is* $F_i(t)$*. Let* $p_{ij} = \Pr\{X_n = j \mid X_{n-1} = i\}$ *be the Markov chain single step transition probabilities. Let* $N(t) = \max\{n : T_n \leq t\}$*, so that* $N(t)$ *is the number of transitions epochs in* $(0, t]$*. Let* $L_j(t) =$ *number of times* $X_n = j$ *for* $0 < n \leq N(t)$*. Thus* $L_j(t)$ *represents the number of times state* j *has been entered in the interval* $(0, t]$*.*

The state entering renewal counting processes are the standard counting processes referred to in the MRP literature, see for example Karr (1986), pg. 344. A slight modification yields the following state exiting counting process definition.

DEFINITION 2. *Let* $(X,T) = \{X_n, T_n; n \geq 0\}$ *be a Markov renewal process as defined in Definition 1. Let* $N(t) = \max\{n : T_n \leq t\}$, *so that* $N(t)$ *is the number of transition epochs in* $(0,t]$. *Let* $N_j(t)$ = *number of times* $X_n = j$ *for* $0 \leq n \leq N(t)-1$ *with* $N_j(t) = 0$ *for all* j *if* $N(t) = 0$. *Thus* $N_j(t)$ *represents the number of times state* j *has been exited in the interval* $(0,t]$. *We call the* $N_j(t)$ *the Exiting Counting Processes (ECP's) associated with the MRP.*

In order to analyze the exiting counting processes, we develop expressions for the moments in the following sections.

2.3. ECP Moment Solutions

In this section, we develop Laplace-Stieltjes transform expressions for the ECP conditional first, second, and cross moments given an initial state $X(0) = i$, and moment solutions for the equilibrium versions of these processes. We also show the relationship between the moments of the state exiting counting processes and the state entering counting processes. We denote the Laplace-Stieltjes transform of a real-valued function $G(\cdot), t \geq 0$, as

$$\tilde{G}(s) = \int_0^\infty e^{-st} dG(t)$$

and $\mathbf{G}(\cdot) = [G_{ij}(\cdot)]$ as a matrix of functions. Also, we let $\mathrm{diag}(d_1, \ldots, d_n)$ represent a matrix with $d_i, i = 1, \ldots, n$ on the diagonal and zeros elsewhere, $\mathbf{I} = diag(1, \ldots, 1)$, and $\mathbf{G}^d = \mathrm{diag}(G_{11}, \ldots, G_{nn})$.

The following Theorem establishes the conditional moments of the state exiting counting processes in terms of Laplace-Stieltjes transforms, and follows directly from an application of renewal equation arguments.

THEOREM 1. *Let* $N_j(t), j = 1, \ldots, K$, *be the ECP's associated with the MRP as defined in Definitions 1 and 2. Let* $M_{ij}(t) = \mathrm{E}[N_j(t) \mid X(0) = i]$, $W_{ij}(t) = \mathrm{E}\left[N_j^2(t) \mid X(0) = i\right]$, *and* $h_{ijk}(t) = \mathrm{E}[N_j(t)N_k(t) \mid X(0) = i]$, *where* $h_{i(jk)}(t) = h_{i(kj)}(t)$ *for* $j \neq k$, *then*

$$\tilde{M}_{ij}(s) = \begin{cases} \tilde{F}_i(s) \sum_{k=1}^K p_{ik} \tilde{M}_{kj}(s) & \text{if } i \neq j \\ \tilde{F}_i(s) \left[1 + \sum_{k=1}^K p_{ik} \tilde{M}_{kj}(s)\right] & \text{if } i = j \end{cases} \tag{2}$$

$$\tilde{W}_{ij}(s) = \begin{cases} \tilde{F}_i(s)\sum_{k=1}^{K} p_{ik}\tilde{W}_{kj}(s) & \text{if } i \neq j \\ \tilde{F}_i(s)\left[1 + \sum_{k=1}^{K} p_{ik}\left(\tilde{W}_{kj}(s) + 2\tilde{M}_{kj}(s)\right)\right] & \text{if } i = j \end{cases} \quad (3)$$

$$\tilde{h}_{i(jk)}(s) = \begin{cases} \tilde{F}_i(s)\sum_{\ell=1}^{K} p_{i\ell}\tilde{h}_{\ell(jk)}(s) & \text{if } i \neq j, i \neq k, j \neq k \\ \tilde{F}_i(s)\left[\sum_{\ell=1}^{K} p_{i\ell}\left(\tilde{h}_{\ell(jk)}(s) + \tilde{M}_{\ell k}(s)\right)\right] & \text{if } i = j \text{ and } j \neq k \end{cases} \quad (4)$$

The proof appears in the appendix.

The following theorem gives the equilibrium moments for the state exiting counting processes. In Section 3, we will utilize the moments to analyze the relationships between the exiting counting processes and to develop asymptotic approximations.

THEOREM 2. *Let $N_j(t), j = 1, \ldots, K$, be the ECP's associated with the MRP as defined in Definitions 1 and 2. Assume that the processes, $N_j(t)$, have existed in the distant past and that we start observing the process at a randomly selected point in time which we define to be $t = 0$. We denote this process as the equilibrium process. Furthermore, suppose that the Markov chain $\{X_n, n \geq 0\}$, associated with $\{X(t), t \geq 0\}$ is ergodic. Let $M_j(t) = \mathrm{E}[N_j(t)]$, $W_j(t) = \mathrm{E}\left[N_j^2(t)\right]$, and $h_{jk}(t) = \mathrm{E}[N_j(t)N_k(t)]$ with $j \neq k$ for the process whose origin starts at the randomly selected point in time. Let $A_i(t) = \Pr\{T_1 \leq t \mid X(0) = i\}$, $\pi_j = \lim_{n\to\infty}\Pr\{X_n = j\}$, $P_{ij}(t) = \Pr\{X(t) = j \mid X(0) = i\}$, and $\nu_j = \lim_{t\to\infty} P_{ij}(t)$ then*

$$\tilde{M}_j(s) = \frac{\nu_j}{\theta_j s} \quad (5)$$

$$\tilde{W}_j(s) = \nu_j\tilde{A}_j(s)\left[1 + 2\sum_{k=1}^{K} p_{jk}\tilde{M}_{kj}(s)\right] + \sum_{i=1}^{K}\nu_i\tilde{A}_i(s)\sum_{k=1}^{K} p_{ik}\tilde{W}_{kj}(s) \quad (6)$$

$$\tilde{h}_{jk}(s) = \sum_{i=1}^{K}\nu_i\tilde{A}_i(s)\sum_{\ell=1}^{K} p_{i\ell}\tilde{h}_{\ell jk}(s) + \nu_j\tilde{A}_j(s)\sum_{\ell=1}^{K} p_{j\ell}\tilde{M}_{\ell k}(s) + \nu_k\tilde{A}_k(s)\sum_{\ell=1}^{K} p_{k\ell}\tilde{M}_{\ell j}(s) \quad (7)$$

where

$$\tilde{A}_i(s) = \frac{1 - \tilde{F}_i(s)}{\theta_i s} \tag{8}$$

The proof appears in the appendix.

We now state the relationship between the conditional moments of the state exiting counting process and the conditional moments of the state entering counting process.

THEOREM 3. *Let $L_j(t)$ and $N_j(t)$ be defined for a MRP as defined in Definitions 1 and 2. Define $Q_{ij}(t) = p_{ij}F_i(t)$. Let $H_{ij}(t) = \mathrm{E}\left[L_j(t) \mid X(0) = i\right]$, and $M_{ij}(t) = \mathrm{E}\left[N_j(t) \mid X(0) = i\right]$, and $B(t) = \mathrm{diag}\left[F_1(t), \ldots, F_K(t)\right]$. Then*

$$\tilde{\mathbf{M}}(s) = \left[\mathbf{I} - \tilde{\mathbf{Q}}(s)\right]^{-1} \tilde{\mathbf{B}}(s) \tag{9}$$

and

$$\tilde{\mathbf{M}}(s) = \left[\tilde{\mathbf{H}}(s) + \mathbf{I}\right] \tilde{\mathbf{B}}(s) \tag{10}$$

The proof appears in the appendix. The following theorem ties the results of Theorem 3 with the results of Theorem 1.

THEOREM 4. *Define $Q_{ij}(t) = p_{ij}F_i(t)$. Let $M_{ij}(t) = \mathrm{E}\left[N_j(t) \mid X(0) = i\right]$, $W_{ij}(t) = \mathrm{E}\left[N_j^2(t) \mid X(0) = i\right]$, and $h_{i(jk)}(t) = \mathrm{E}\left[N_j(t)N_k(t) \mid X(0) = i\right]$, where $h_{i(jk)}(t) = h_{i(kj)}(t)$ for $j \neq k$, and $B(t) = \mathrm{diag}\left(F_1(t), \ldots, F_K(t)\right)$. Let $\vec{\mathbf{h}}^{(jk)}(t) = \left[h_{1(jk)}(t), h_{2(jk)}(t), \ldots, h_{K(jk)}(t)\right]$ and let $\vec{\mathbf{m}}^{(jk)}(t)$ be a $n \times 1$ vector with elements*

$$m_i = \begin{cases} 0 & \text{if } i \neq j \text{ and } i \neq k \\ M_{jk}(t) & \text{if } i = j \\ M_{kj}(t) & \text{if } i = k \end{cases}$$

Then

$$\tilde{\mathbf{W}}(s) = \tilde{\mathbf{M}}(s)\left\{2\left[\tilde{\mathbf{M}}(s)\tilde{\mathbf{B}}^{-1}(s)\right]^d - \mathbf{I}\right\} \tag{11}$$

$$\tilde{\vec{\mathbf{h}}}^{(jk)}(s) = \tilde{\mathbf{M}}(s)\tilde{\mathbf{B}}^{-1}(s)\tilde{\vec{\mathbf{m}}}^{(jk)}(s) \tag{12}$$

From Theorems 1 and 2, we see that the moment expressions can be quite complicated in general. In order to be able to utilize moment expressions in the analysis of multi-product class production systems, we develop asymptotic expansions for the moments of the ECP's in terms of the moments of $F_i(t)$ and the transition probabilities of the embedded Markov chain. The next section presents asymptotic expansions and defines approximations for the equilibrium moments of the ECP's.

3. Asymptotic Moment Expansions

Throughout the remainder of the paper, we assume that we are dealing with a two-state irreducible MRP with ergodic states, i.e. $0 < p_{ii} < 1$, for i= 1,2. Although we concentrate on the two state case, we note that by using our Theorem 3 along with the matrix methodology and results presented in Hunter (1969) a generalization could be developed for more than two states. We feel that for the purposes of this paper that it is sufficient to fully illustrate the two state case. We will first derive asymptotic expansions for the equilibrium versions of $W_1(t), W_2(t)$, and $h_{12}(t)$. We will then develop the exact expressions for the case where $F_i(t)$ is negative exponential. Finally, we will propose and evaluate two approximations for the equilibrium moments.

We begin by developing expressions for the mean recurrence time and the variance of the recurrence time for the two state process.

LEMMA 1. *Let (X, T) be a Markov renewal process as defined in Definition 1 having state space, $S = \{1, 2\}$, with $0 < p_{ii} < 1, i = 1, 2$. Let $F_i(t) = \Pr\{Y_n \leq t \mid X_{n-1} = i\}$, and let θ_i and δ_i^2 be the mean and the variance for $F_i(t)$. Let α_i, σ_i^2 be the mean and variance of the recurrence time for states $i = 1, 2$. Then*

$$\alpha_1 = \frac{\pi_1\theta_1 + \pi_2\theta_2}{\pi_1} \tag{13}$$

$$\alpha_2 = \frac{\pi_1\theta_1 + \pi_2\theta_2}{\pi_2} \tag{14}$$

$$\sigma_1^2 = \frac{\pi_1\delta_1^2 + \pi_2\delta_2^2}{\pi_1} + \frac{\pi_2\theta_2^2}{\pi_1^2}\left(\frac{2}{\beta} - 1\right) \tag{15}$$

$$\sigma_2^2 = \frac{\pi_1\delta_1^2 + \pi_2\delta_2^2}{\pi_2} + \frac{\pi_1\theta_1^2}{\pi_2^2}\left(\frac{2}{\beta} - 1\right) \tag{16}$$

where

$$\beta = 2 - p_{11} - p_{22} \tag{17}$$

The proof appears in the appendix.

The following lemma establishes the asymptotic expansions for the conditional first moments of the ECP's and will be utilized in establishing the asymptotic expansions for the equilibrium ECP's.

LEMMA 2. *Let (X,T) be a Markov renewal process as defined in Definition 1 having state space, $S = \{1,2\}$, with $0 < p_{ii} < 1, i = 1,2$. Let $M_{ij}(s)$ be given as in Theorem 1. Let α_i, σ_i^2 be the mean and variance of the recurrence time for states $i = 1,2$. Then*

$$\tilde{M}_{11}(s) = \frac{1}{\alpha_1 s} + \left(\frac{\sigma_1^2 - \alpha_1^2}{2\alpha_1^2} + \nu_2\right) + c_{11}s + o(s) \tag{18}$$

$$\tilde{M}_{12}(s) = \frac{1}{\alpha_2 s} + \left(\frac{\sigma_2^2 - \alpha_2^2}{2\alpha_2^2} - \frac{\theta_1}{\pi_1\alpha_1\beta} + \nu_1\right) + c_{12}s + o(s) \tag{19}$$

$$\tilde{M}_{21}(s) = \frac{1}{\alpha_1 s} + \left(\frac{\sigma_1^2 - \alpha_1^2}{2\alpha_1^2} - \frac{\theta_2}{\pi_2\alpha_2\beta} + \nu_2\right) + c_{21}s + o(s) \tag{20}$$

$$\tilde{M}_{22}(s) = \frac{1}{\alpha_2 s} + \left(\frac{\sigma_2^2 - \alpha_2^2}{2\alpha_2^2} + \nu_1\right) + c_{22}s + o(s) \tag{21}$$

Formulas for obtaining c_{11}, c_{12}, c_{21}, and c_{22} are given in the appendix because of the length of the expressions; see Equation 95.

The proof, which appears in the appendix, is similar to those given in Kshirsagar and Gupta (1967), pp. 598–599 and Hunter (1969) and involves considerable algebraic manipulations.

3.1. Asymptotic Moment Expansions

Substituting the $\tilde{M}_{ij}(s)$ given by Lemma 2 into the two state equations given by Theorem 2 and then collecting terms and inverting yields the following result for the equilibrium ECP's.

THEOREM 5. *Let (X,T) be a Markov renewal process as defined in Definition 1 having state space, $S = \{1,2\}$, with $0 < p_{ii} < 1, i = 1,2$. Let $M_j(t), W_j(t)$, and $h_{jk}(t)$ be as defined in Theorem 2. Then*

$$M_1(t) = t/\alpha_1 \tag{22}$$

$$M_2(t) = t/\alpha_2 \tag{23}$$

$$W_1(t) = B_1 + B_2 t + B_3 t^2/2 + o(1) \tag{24}$$

$$W_2(t) = D_1 + D_2 t + D_3 t^2/2 + o(1) \tag{25}$$

$$h_{12}(t) = A_1 + A_2 t + A_3 t^2/2 + o(1) \tag{26}$$

where

$$B_2 = \sigma_1^2/\alpha_1^3 \tag{27}$$

$$B_3 = 2/\alpha_1^2 \tag{28}$$

$$D_2 = \sigma_2^2/\alpha_2^3 \tag{29}$$

$$D_3 = 2/\alpha_2^2 \tag{30}$$

$$A_2 = \frac{\pi_1\pi_2}{\theta^3}\left[\pi_1\delta_1^2 + \pi_2\delta_2^2 - \theta_1\theta_2\left(\frac{2}{\beta}-1\right)\right] \tag{31}$$

$$A_3 = 2/(\alpha_1\alpha_2) \tag{32}$$

$$\theta = \pi_1\theta_1 + \pi_2\theta_2 \tag{33}$$

The coefficients B_1, D_1, and A_1 are given in Equations 105– 107 of the appendix because of their length.

The proof appears in the appendix. The coefficients B_1, D_1, and A_1 are complicated functions of the p_{ij} and the first three moments of $F_i(t)$. One of our approximations involves dropping the coefficients B_1, D_1, and A_1 since they would be small in comparison to the other terms for large t. To evaluate the accuracy of this approximation we derive exact analytical expressions for the case of negative exponential sojourn time distributions.

3.2. Negative Exponential Moment Solutions

In this section, we present the exact analytical expressions of the moments of the equilibrium process for the two state MRP with $F_i(t) = 1 - e^{-t/\theta_i}, i = 1, 2$.

COROLLARY 1. *Let (X, T) be a Markov renewal process as defined in Definition 1 having state space, $S = \{1, 2\}$, with $0 < p_{ii} < 1, i = 1, 2$. Let $N_j(t), j = 1, \ldots, K$, be the ECP's associated with the MRP as defined in Definition 2. Let $F_i(t) = 1 - e^{-t/\theta_i}, i = 1, 2$. Then*

$$\mathrm{E}\left[N_1(t)\right] = t/\alpha_1 \tag{34}$$

$$\mathrm{E}\left[N_2(t)\right] = t/\alpha_2 \tag{35}$$

$$\mathrm{E}\left[N_1^2(t)\right] = \frac{t^2}{\alpha_1^2} + \frac{\sigma_1^2\,t}{\alpha_1^3} + \left(\frac{\alpha_1^2-\sigma_1^2}{\alpha_1^3\gamma}\right)\left(1-e^{-\gamma t}\right) \tag{36}$$

$$\mathrm{E}\left[N_2^2(t)\right] = \frac{t^2}{\alpha_2^2} + \frac{\sigma_2^2\,t}{\alpha_2^3} + \left(\frac{\alpha_2^2-\sigma_2^2}{\alpha_2^3\gamma}\right)\left(1-e^{-\gamma t}\right) \tag{37}$$

$$\mathrm{E}\left[N_1(t)N_2(t)\right] = \frac{t^2}{\alpha_1\alpha_2} + \frac{1}{\alpha_1\alpha_2}\left[\frac{\delta^2}{\theta} - \frac{\theta_1\theta_2}{\theta}\left(\frac{2}{\beta}-1\right)\right] t + \frac{-1}{\alpha_1\alpha_2\gamma}\left[\frac{\delta^2}{\theta} - \frac{\theta_1\theta_2}{\theta}\left(\frac{2}{\beta}-1\right)\right]\left(1-e^{-\gamma t}\right) \tag{38}$$

$$\mathrm{Var}\left[N_1(t)\right] = \frac{\sigma_1^2\,t}{\alpha_1^3} + \left(\frac{\alpha_1^2-\sigma_1^2}{\alpha_1^3\gamma}\right)\left(1-e^{-\gamma t}\right) \tag{39}$$

$$\mathrm{Var}\left[N_2(t)\right] = \frac{\sigma_2^2\,t}{\alpha_2^3} + \left(\frac{\alpha_2^2-\sigma_2^2}{\alpha_2^3\gamma}\right)\left(1-e^{-\gamma t}\right) \tag{40}$$

$$\begin{aligned}
\mathrm{Cov}\left[N_1(t), N_2(t)\right] &= \frac{1}{\alpha_1\alpha_2}\left[\frac{\delta^2}{\theta} - \frac{\theta_1\theta_2}{\theta}\left(\frac{2}{\beta} - 1\right)\right] t + \\
&\quad \frac{-1}{\alpha_1\alpha_2\gamma}\left[\frac{\delta^2}{\theta} - \frac{\theta_1\theta_2}{\theta}\left(\frac{2}{\beta} - 1\right)\right]\left(1 - e^{-\gamma t}\right) \quad (41) \\
\theta &= \pi_1\theta_1 + \pi_2\theta_2 \quad (42) \\
\delta^2 &= \pi_1\delta_1^2 + \pi_2\delta_2^2 \quad (43) \\
\gamma &= \theta\beta/\theta_1\theta_2 \quad (44)
\end{aligned}$$

The proof appears in the appendix.

4. Moment Approximations

In this section, we present two potential approximations for the moments of the state exiting counting processes for the two state MRP. We evaluate the approximations both analytically and empirically in order to justify the use of the approximations. We define the absolute error(AE) and absolute relative error(ARE) of the approximation as

$$\begin{aligned}
\mathrm{AE}_j(\cdot) &= |\mathrm{true} - \mathrm{approximation}| \quad (45) \\
\mathrm{ARE}_j(\cdot) &= \frac{|\mathrm{true} - \mathrm{approximation}|}{|\mathrm{true}|} \quad (46)
\end{aligned}$$

where j refers to the relevant approximation.

We now define two approximations for exiting counting process moments.

DEFINITION 3. *Let the conditions of Theorem 5 hold. The moments* $\mathrm{E}\left[N_1^2(t)\right]$, $\mathrm{E}\left[N_2^2(t)\right]$, *and* $\mathrm{E}\left[N_1(t)N_2(t)\right]$ *have the form* $k_0 + k_1 t + k_2 t^2/2 + o(1)$. *We define Approximation 1 by dropping the* $o(1)$ *terms so that*

$$\begin{aligned}
\mathrm{E}\left[N_1^2(t)\right] &\doteq B_1 + B_2 t + B_3 t^2/2 \quad (47) \\
\mathrm{E}\left[N_2^2(t)\right] &\doteq D_1 + D_2 t + D_3 t^2/2 \quad (48) \\
\mathrm{E}\left[N_1(t)N_2(t)\right] &\doteq A_1 + A_2 t + A_3 t^2/2 \quad (49)
\end{aligned}$$

DEFINITION 4. *Let the conditions of Theorem 5 hold. The moments* $\mathrm{E}\left[N_1^2(t)\right]$, $\mathrm{E}\left[N_2^2(t)\right]$, *and* $\mathrm{E}\left[N_1(t)N_2(t)\right]$ *have the form* $k_0 + k_1 t + k_2 t^2/2 + o(1)$. *We define Approximation 2 by dropping the* k_0 *and* $o(1)$ *terms so that*

$$\begin{aligned}
\mathrm{E}\left[N_1^2(t)\right] &\doteq B_2 t + B_3 t^2/2 \quad (50) \\
\mathrm{E}\left[N_2^2(t)\right] &\doteq D_2 t + D_3 t^2/2 \quad (51) \\
\mathrm{E}\left[N_1(t)N_2(t)\right] &\doteq A_2 t + A_3 t^2/2 \quad (52)
\end{aligned}$$

4.1. Analytical Evaluation of Approximations

The purpose of this section is to give some analytical properties of the approximations for the negative exponential sojourn distribution function case.

COROLLARY 2. *Let the conditions of Corollary 1 hold. The moments* $\mathrm{E}\left[N_1^2(t)\right]$, $\mathrm{E}\left[N_2^2(t)\right]$, *and* $\mathrm{E}\left[N_1(t)N_2(t)\right]$ *have the form* $k_0(1-e^{-\gamma t}) + k_1 t + k_2 t^2/2$. *For Approximation 1, we have for fixed parameters that* $\mathrm{AE}_1(\cdot)$ *has the following form*

$$\mathrm{AE}_1(\cdot) = \left|k_0 e^{-\gamma t}\right|$$

so that

	$\mathrm{E}\left[N_1^2(t)\right]$	$\mathrm{E}\left[N_2^2(t)\right]$	$\mathrm{E}\left[N_1(t)N_2(t)\right]$
$\lim_{t\to+\infty} \mathrm{AE}_1(\cdot)$	0	0	0
$\lim_{t\to 0^+} \mathrm{AE}_1(\cdot)$	$\lvert k_0\rvert$	$\lvert k_0\rvert$	$\lvert k_0\rvert$
$\lim_{t\to+\infty} \mathrm{ARE}_1(\cdot)$	0	0	0
$\lim_{t\to 0^+} \mathrm{ARE}_1(\cdot)$	$+\infty$	$+\infty$	$+\infty$

and for fixed t, $\mathrm{AE}_1(\cdot)$ *and* $\mathrm{ARE}_1(\cdot)$ *may be arbitrarily large depending on the choice of parameters.*

The proofs of $\mathrm{AE}_1(\cdot)$ for ($t \to 0^+$, and $t \to +\infty$) follow immediately from the functional form of $\mathrm{AE}_1(\cdot)$. The proof of $\mathrm{ARE}_1(\cdot)$ for ($t \to +\infty$) follows from an application of L'Hospital's rule, and the proof of $\mathrm{ARE}_1(\cdot)$ for ($t \to 0^+$) follows from the functional form of $\mathrm{ARE}_1(\cdot)$. The result for fixed t follows by noting that ($k_0 \to \infty$) as ($p_{11} \to 1$ and $p_{22} \to 1$).

COROLLARY 3. *Let the conditions of Corollary 1 hold. The moments* $\mathrm{E}\left[N_1^2(t)\right]$, $\mathrm{E}\left[N_2^2(t)\right]$, *and* $\mathrm{E}\left[N_1(t)N_2(t)\right]$ *have the form* $k_0(1-e^{-\gamma t}) + k_1 t + k_2 t^2/2$. *For Approximation 2, we have for fixed parameters that* $\mathrm{AE}_2(\cdot)$ *has the following form*

$$\mathrm{AE}_2(\cdot) = \left|k_0(1-e^{-\gamma t})\right|$$

so that

	$\mathrm{E}\left[N_1^2(t)\right]$	$\mathrm{E}\left[N_2^2(t)\right]$	$\mathrm{E}\left[N_1(t)N_2(t)\right]$
$\lim_{t\to+\infty} \mathrm{AE}_2(\cdot)$	$\lvert k_0\rvert$	$\lvert k_0\rvert$	$\lvert k_0\rvert$
$\lim_{t\to 0^+} \mathrm{AE}_2(\cdot)$	0	0	0
$\lim_{t\to+\infty} \mathrm{ARE}_2(\cdot)$	0	0	0
$\lim_{t\to 0^+} \mathrm{ARE}_2(\cdot)$	$\left\lvert 1-\frac{\sigma_1^2}{\alpha_1^2}\right\rvert$	$\left\lvert 1-\frac{\sigma_2^2}{\alpha_2^2}\right\rvert$	$+\infty$

and for fixed t, $\mathrm{AE}_2(\cdot)$ *and* $\mathrm{ARE}_2(\cdot)$ *may be arbitrarily large depending on the choice of parameters.*

The proofs of $\mathrm{AE}_2(\cdot)$ for ($t \to 0^+$, and $t \to +\infty$) follow immediately from the functional form of $\mathrm{AE}_2(\cdot)$. The proofs of $\mathrm{ARE}_2(\cdot)$ for ($t \to +\infty$) and ($t \to 0^+$) follow from an application of L'Hospital's rule. The result for fixed t follows by noting that ($k_0 \to \infty$) as ($p_{11} \to 1$ and $p_{22} \to 1$).

Remark. If we define, $\mathrm{ARE}_1(\cdot)$ and $\mathrm{ARE}_2(\cdot)$ as the absolute relative error for their respective approximations, then $\mathrm{ARE}_2(\cdot) = \left|(e^{\gamma t} - 1)\right| \mathrm{ARE}_1(\cdot)$. Note that both $\mathrm{ARE}_1(\cdot)$ and $\mathrm{ARE}_2(\cdot)$ approach zero as t approaches positive infinity.

Figure 1 illustrates the behavior of the approximations compared to the true function for the case ($\theta_1 = 20$, $\theta_2 = 10$, $p_{11} = p_{22} = 0.9$). Approximation 1 approaches the true function as time increases while Approximation 2 approaches a fixed constant distance from the true; however, Figure 2 illustrates that in terms of absolute relative error Approximation 2 becomes competitive with Approximation 1 very quickly. In the next section, we empirically examine the accuracy of the approximations over a range of parameter values.

$\theta_1 = 20$, $\theta_2 = 10$, $p_{11} = p_{22} = 0.9$

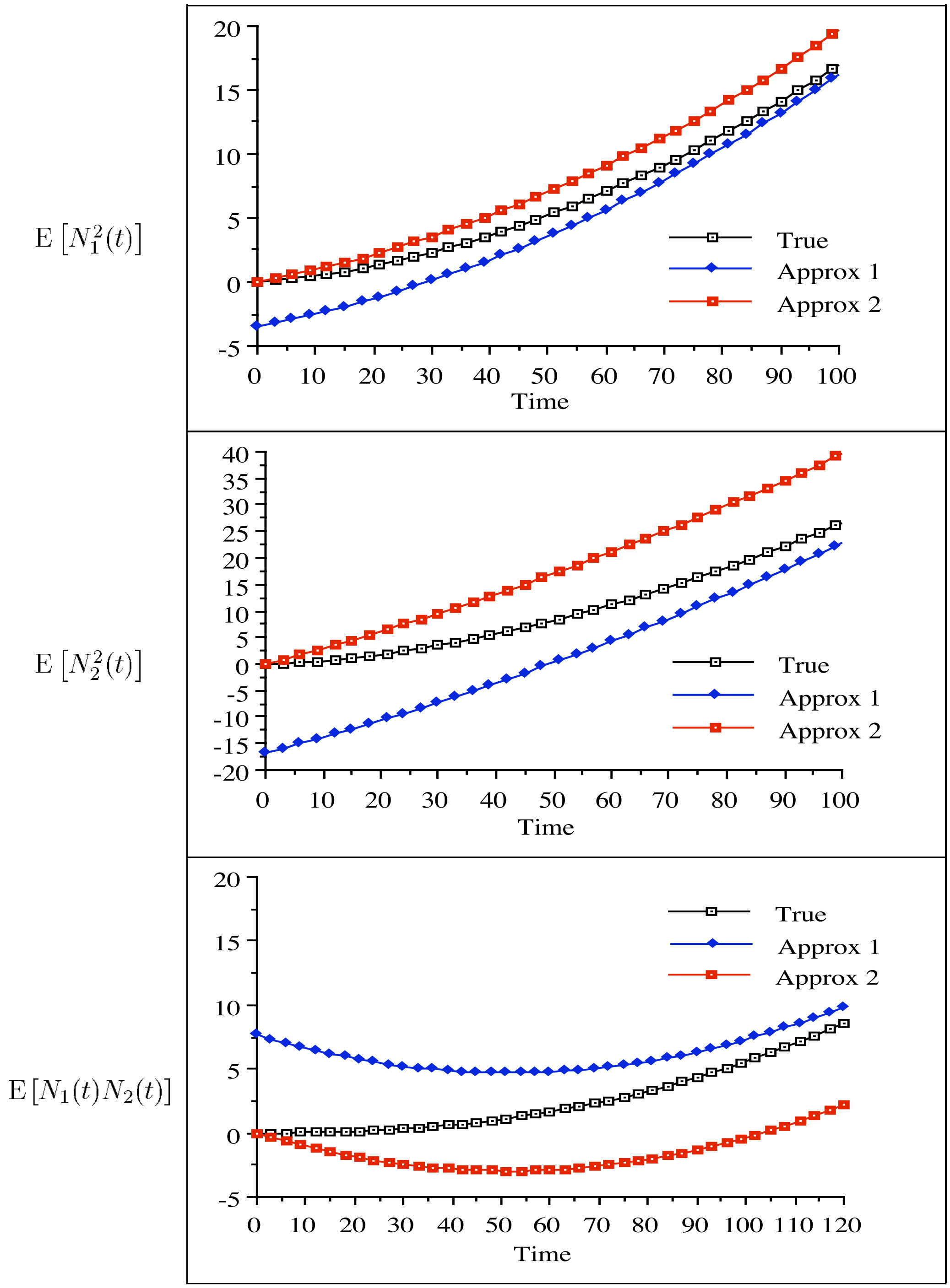

Figure 1. Example Function Plots

$\theta_1 = 20$, $\theta_2 = 10$, $p_{11} = p_{22} = 0.9$

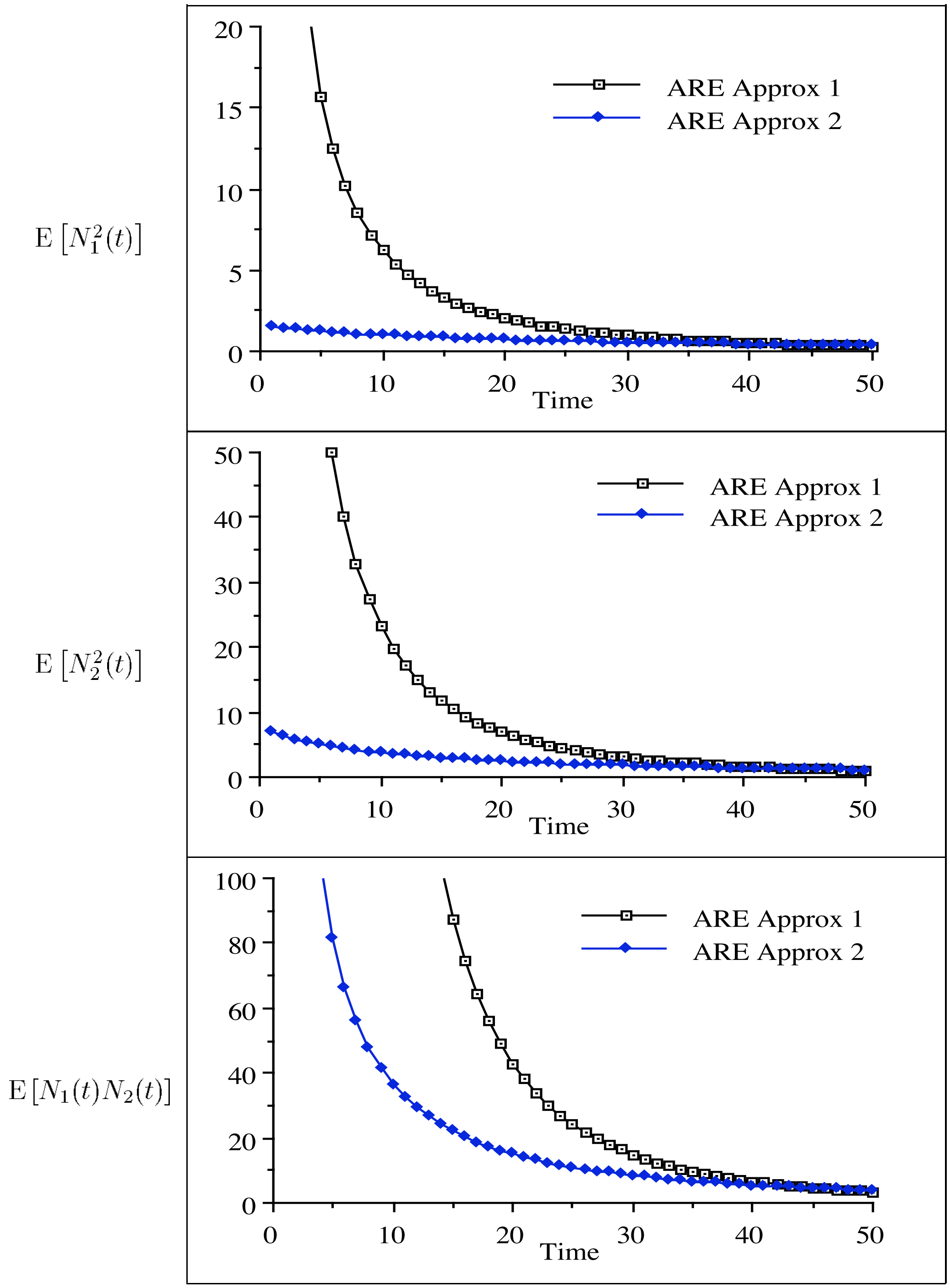

Figure 2. Example Absolute Relative Error Plots

4.2. Empirical Evaluation of Approximations

In this section, we examine empirically the performance of the approximations for the negative exponential case. For the negative exponential case the parameters to vary are θ_1,θ_2, p_{11}, and p_{22}. We randomly select $n = 10000$ parameter values according to a uniform distribution such that the range of parameters is $0 < p_{11}, p_{22} < 1$ and $1 < \theta_1, \theta_2 < 100$ and compute $\text{ARE}_1(\cdot)$ and $\text{ARE}_2(\cdot)$. Tables I and II report the statistics for sampled absolute relative errors for $t = 1, 60, 120, 240, 480$. Tables I and II indicate that in terms of absolute relative error the approximations are quite good over a reasonable range of parameters when t is large compared to the parameters. Approximation 2 is simpler in form and allows the possibility of easier analytical analysis. Based upon the analytical and empirical evaluations of the approximations, we would recommend Approximation 2 because of its simpler form and because it only depends upon the first two moments of $F_i(t)$. In the next section, we will illustrate the use of the approximations in two simple but realistic examples.

Table I. Sample Results for $\mathrm{ARE}_1(\cdot)$.

t		$\mathrm{ARE}(\mathrm{E}\left[N_1^2(t)\right])$	$\mathrm{ARE}(\mathrm{E}\left[N_2^2(t)\right])$	$\mathrm{ARE}(\mathrm{E}\left[N_1(t)N_2(t)\right])$
1	$\bar{x}$	88.58	58.84	1.30×10^5
	s^2/n	5.99×10^3	998.43	1.36×10^{10}
	n	10000	10000	10000
	min	1.65×10^{-5}	5.48×10^{-6}	6.27×10^{-5}
	max	7.73×10^5	2.22×10^5	1.16×10^5
60	$\bar{x}$	0.081	0.127	11.219
	s^2/n	4.12×10^{-3}	7.04×10^{-3}	89.85
	n	10000	10000	10000
	min	0	0	0
	max	630.59	718.02	9.31×10^4
120	$\bar{x}$	0.02	0.033	1.631
	s^2/n	2.55×10^{-4}	4.91×10^{-4}	1.831
	n	10000	10000	10000
	min	0	0	0
	max	155.57	192.42	13215.3
240	$\bar{x}$	0.0048	0.0078	0.2138
	s^2/n	1.45×10^{-5}	2.97×10^{-5}	0.0309
	n	10000	10000	10000
	min	0	0	0
	max	36.9	47.87	1707.57
480	$\bar{x}$	0.001	0.0017	0.0251
	s^2/n	7.1×10^{-7}	1.51×10^{-6}	4.28×10^{-4}
	n	10000	10000	10000
	min	0	0	0
	max	8.19	10.95	200.76

where $\bar{x}$ is the sample average and s^2 is the sample variance.

Table II. Sample Results for $\mathrm{ARE}_2(\cdot)$.

t		ARE(E $[N_1^2(t)]$)	ARE(E $[N_2^2(t)]$)	ARE(E $[N_1(t)N_2(t)]$)
1	$\bar{x}$	0.5946	0.5897	99.17
	s^2/n	0.0024	0.0011	4.84×10^{3}
	n	10000	10000	10000
	min	1.59×10^{-5}	6.25×10^{-6}	8.57×10^{-5}
	max	457.73	195.56	6.86×10^{5}
60	$\bar{x}$	0.016	0.1894	0.4902
	s^2/n	6.26×10^{-6}	1.39×10^{-5}	0.1225
	n	10000	10000	10000
	min	4.21×10^{-8}	2.91×10^{-8}	4.12×10^{-8}
	max	22.78	25.94	3364.44
120	$\bar{x}$	0.0065	0.0081	0.1457
	s^2/n	1.61×10^{-6}	3.91×10^{-5}	0.0105
	n	10000	10000	10000
	min	1.09×10^{-8}	7.4×10^{-9}	1.03×10^{-8}
	max	11.44	14.15	972.19
240	$\bar{x}$	0.0026	0.0034	0.0398
	s^2/n	3.9×10^{-7}	9.9×10^{-7}	7.74×10^{-4}
	n	10000	10000	10000
	min	2.8×10^{-9}	1.9×10^{-9}	2.6×10^{-9}
	max	5.63	7.30	260.48
480	$\bar{x}$	0.001	0.001	0.01
	s^2/n	9.0×10^{-8}	2.3×10^{-7}	5.03×10^{-5}
	n	10000	10000	10000
	min	7.0×10^{-10}	5.0×10^{-10}	6.0×10^{-10}
	max	2.69	3.59	65.92

where $\bar{x}$ is the sample average and s^2 is the sample variance.

5. Example Applications

In this section, we present two examples to illustrate the results developed within this paper.

EXAMPLE 1. The purpose of this example is to show the effect of attempting to group the production of the products by type. Referring to the example of Section 2, suppose an order for 450 type 1 and 450 type 2 products was to be delivered two weeks hence, where two weeks is (2×5 days $\times 3$ shifts $\times 480$ minutes = 14,400 minutes). If we could set up the Markov chain transition matrix to best achieve the order, what would it be? Note that the example assumes that there is no setup time. Using the results for the negative exponential distribution given in Corollary 1, Table III presents analytical results for a shift given two possible Markov chain configurations.

Table III. Example 1 Analytical Results

$\tau = 14,400$ minutes $\theta_1 = 20$ minutes $\theta_2 = 10$ minutes

	Case	
	1	2
(p_{11}, p_{22})	$(0.9, 0.9)$	$(0.1, 0.1)$
$\mathrm{E}[N_1(\tau)]$	480.0	480.0
$\mathrm{E}[N_2(\tau)]$	480.0	480.0
$\mathrm{Var}[N_1(\tau)]$	1223.209	278.622
$\mathrm{Var}[N_2(\tau)]$	4089.876	314.159
$\mathrm{Corr}[N_1(\tau), N_2(\tau)]$	-0.7358	+0.8208

Case 1 has high return probabilities, (p_{ii}), which could represent the situation where the system produces many of one type and then switches to the other type. Case 2 is close to an alternating renewal process. Both cases are above the target in expected number produced. Without too much thought one might consider Case 1 to be superior since many production systems produce many of one type and then switch to the other; however, in terms of the variability, Case 2 can be argued to be more reliable. By more reliable, we mean more likely to meet the target production. For example, if we assume for simplicity that the production counts have a bivariate normal distribution then we can make a probability statement of the following form,

$$p = \Pr\{N_1(\tau) \geq n_1, N_2(\tau) \geq n_2\}$$

which gives the probability of meeting or exceeding the target level. Based on the data given in Table III and using the bivariate normal assumption with $n_1 = 450$ and $n_2 = 450$, the value of p for Case 1 is 0.489 and the value of p for Case 2 is 0.9388. We see that Case 2 is indeed more reliable. The bivariate normal distribution assumption is reasonable based upon asymptotic results, see for example Taga (1963). Of course, a real production system will have costs associated with switching product types and thus Case 2 may become prohibitive in terms of other factors.

EXAMPLE 2. Referring again to the example of Section 2, suppose we are interested in estimating the mean of the service time distribution from only count data. In Rossetti and Clark (1995), we propose the following linear model

$$Y_i = \sum_{k=1}^{K} X_{ik}\theta_k + \epsilon_i, \; (i = 1, \ldots, n) \tag{53}$$

where K is the number of states, n is the number of observed fixed length time intervals, $Y_i = \tau$ is the amount of time in interval i, X_{ik} is the number of times state k is exited during interval i, θ_k is the mean of the service time distribution function, and ϵ_i is the model error term. We then use the moment functions to analytically evaluate the least squares estimators for θ_i in terms of asymptotic statistical properties and to also develop a competing method of moments estimator.

Other interesting example applications exist such as using the variance function results to plan how many shifts of count observations would be necessary to achieve a desired statistical precision when estimating E$[N_i(\tau)]$.

6. Summary and Conclusions

This paper considered the development of Laplace-Stieltjes transforms for the moments of the number of times a given state is exited in a Markov renewal process, and evaluated asymptotic approximations for the moment functions.

The contributions include:

1. The definition of the state exiting counting process.
2. The analytical formulation of the solutions for the conditional first, second, and covariance moments of the ECP's, and the formulation of the solutions for the unconditional equilibrium versions of the moments.
3. The demonstration of the relationship between the state exiting and state entering counting processes for the conditional first moments.
4. Asymptotic expansions for the conditional first moments of the ECP's for the two state MRP.
5. Asymptotic expansions for the unconditional first, second, and covariance moments of the ECP's for the two state equilibrium case.
6. Analytical and empirical evaluation of two approximations for the unconditional first, second, and covariance moments of the ECP's for the two state equilibrium MRP.

Our evaluation of the two proposed asymptotic approximations indicated that the relative error properties of Approximation 2 were comparative to Approximation 1, and Approximation 2 has a simpler analytical form. The results can be utilized in a variety of decision making processes including the analytical evaluation of more complex stochastic processes with an underlying MRP structure.

Finally, we note that while our illustrative analysis was limited to the two state MRP the results can be applicable to larger state space situations by collapsing the state space down to two states or the results could be generalized by methods similar to that in Hunter (1969). Further analytical and empirical research should be done to examine the affect of the aggregation process on the approximation's performance.

7. Appendix

Proofs of the lemmas, theorems, and corollaries are given in this appendix.

Proof for Theorem 1:

The proof uses renewal equation arguments. If $X_0 = i$ then the distribution function for T_1 is $F_i(t)$, condition on the first event so that

$$\mathrm{E}\left[N_j(t) \mid X(0) = i\right] = \int_0^{\infty} \mathrm{E}\left[N_j(t) \mid X(0) = i, T_1 = t_1\right] dF_i(t_1) \quad (54)$$

Consider $T_1 = t_1 > t$ then $N_j(t) = 0$, thus,

$$\mathrm{E}\left[N_j(t) \mid X(0) = i, T_1 = t_1 > t\right] = 0 \tag{55}$$

Consider $0 \le T_1 = t_1 \le t$, for $i \ne j$,

$$\begin{aligned} &\mathrm{E}\left[N_j(t) \mid X(0) = i, T_1 = t_1 \le t\right] \\ &= \sum_{k=1}^{K} \mathrm{E}\left[N_j(t) \mid X(t_1) = k\right] \Pr\left\{X(t_1) = k \mid X(0) = i\right\} \\ &= \sum_{k=1}^{K} M_{kj}(t - t_1) p_{ik} \end{aligned} \tag{56}$$

Using Equation 54, for $i \ne j$, the result is

$$\mathrm{E}\left[N_j(t) \mid X(0) = i\right] = \sum_{k=1}^{K} p_{ik} \int_0^t M_{kj}(t - t_1) dF_i(t_1) \tag{57}$$

Consider $0 \le T_1 = t_1 \le t$. For $i = j$ we have that

$$\begin{aligned} &\mathrm{E}\left[N_j(t) \mid X(0) = i, T_1 = t_1 \le t\right] \\ &= \sum_{k=1}^{K} \mathrm{E}\left[N_j(t) + 1 \mid X(t_1) = k\right] \Pr\left\{X(t_1) = k \mid X(0) = i\right\} \\ &= 1 + \sum_{k=1}^{K} M_{kj}(t - t_1) p_{ik} \end{aligned} \tag{58}$$

Using Equation 54, when $i = j$,

$$\mathrm{E}\left[N_j(t) \mid X(0) = i\right] = F_i(t) + \sum_{k=1}^{K} p_{ik} \int_0^t M_{kj}(t - t_1) dF_i(t_1) \tag{59}$$

The Laplace-Stieltjes transform of Equations 57 and 59 yields the desired result. For the second moment case, condition on the first event, so that,

$$\mathrm{E}\left[N_j^2(t) \mid X(0) = i\right] = \int_0^\infty \mathrm{E}\left[N_j^2(t) \mid X(0) = i, T_1 = t_1\right] dF_i(t_1) \tag{60}$$

Consider $T_1 = t_1 > t$. Then $N_j(t) = 0$ and $N_j^2(t) = 0$, thus,

$$\mathrm{E}\left[N_j^2(t) \mid X(0) = i, T_1 = t_1 > t\right] = 0 \tag{61}$$

Consider $0 \leq T_1 = t_1 \leq t$. For $i \neq j$,

$$\mathrm{E}\left[N_j^2(t) \mid X(0) = i, T_1 = t_1 \leq t\right]$$

$$= \sum_{k=1}^{K} \mathrm{E}\left[N_j^2(t) \mid X(t_1) = k\right] \Pr\{X(t_1) = k \mid X(0) = i\}$$

$$= \sum_{k=1}^{K} W_{kj}(t - t_1) p_{ik} \tag{62}$$

Using Equation 60, the result for $i \neq j$ is

$$\mathrm{E}\left[N_j^2(t) \mid X(0) = i\right] = \sum_{k=1}^{K} p_{ik} \int_0^t W_{kj}(t - t_1) dF_i(t_1) \tag{63}$$

Consider $0 \leq T_1 = t_1 \leq t$. For $i = j$, we have that

$$\mathrm{E}\left[N_j^2(t) \mid X(0) = i, T_1 = t_1 \leq t\right]$$

$$= \sum_{k=1}^{K} \mathrm{E}\left[(N_j(t - t_1) + 1)^2 \mid X(0) = k\right] p_{ik} \tag{64}$$

$$= 1 + \sum_{k=1}^{K} (W_{kj}(t - t_1) + 2M_{kj}(t - t_1)) p_{ik} \tag{65}$$

Using Equation 60, for $i = j$ yields

$$\mathrm{E}\left[N_j^2(t) \mid X(0) = i\right] = F_i(t) + \sum_{k=1}^{K} p_{ik} \int_0^t (W_{kj}(t-t_1) + 2M_{kj}(t-t_1)) dF_i(t_1) \tag{66}$$

The Laplace-Stieltjes transform of Equations 63 and 66 yields the desired result. For the cross moment case, again condition on the first event so that

$$\mathrm{E}\left[N_j(t) N_k(t) \mid X(0) = i\right] = \int_0^\infty \mathrm{E}\left[N_j(t) N_k(t) \mid X(0) = i, T_1 = t_1\right] dF_i(t_1) \tag{67}$$

Consider $T_1 = t_1 > t$. Then $N_j(t) = 0$ and $N_k(t) = 0$, thus,

$$\mathrm{E}\left[N_j(t) N_k(t) \mid X(0) = i, T_1 = t_1 > t\right] = 0 \tag{68}$$

Consider $0 \leq T_1 = t_1 \leq t$. For $i \neq j$, $i \neq k$, and $j \neq k$

$$\mathrm{E}\left[N_j(t) N_k(t) \mid X(0) = i, T_1 = t_1 \leq t\right]$$

$$= \sum_{\ell=1}^{K} \mathrm{E}\left[N_j(t) N_k(t) \mid X(t_1) = \ell\right] \Pr\{X(t_1) = \ell \mid X(0) = i\}$$

$$= \sum_{\ell=1}^{K} h_{\ell jk}(t-t_1)p_{i\ell} \tag{69}$$

Using Equation 67 implies that for $i \neq j$, $i \neq k$, and $j \neq k$,

$$\mathrm{E}\left[N_j(t)N_k(t) \mid X(0)=i\right] = \sum_{\ell=1}^{K} p_{i\ell} \int_0^t h_{\ell jk}(t-t_1)dF_i(t_1) \tag{70}$$

Consider $0 \leq T_1 = t_1 \leq t$. For $i = j$ and $j \neq k$. Specifically, let $i = j$, the argument is the same for the case $i = k$ with an appropriate change of subscripts. We have that

$$\mathrm{E}\left[N_j(t)N_k(t) \mid X(0)=i, T_1 = t_1 \leq t\right]$$

$$= \sum_{\ell=1}^{K} \mathrm{E}\left[(N_j(t-t_1)+1)N_k(t-t_1) \mid X(0)=\ell\right] p_{i\ell} \tag{71}$$

$$= \sum_{\ell=1}^{K} (h_{\ell jk}(t-t_1) + M_{\ell k}(t-t_1))p_{i\ell} \tag{72}$$

Using Equation 67 implies that for $i = j$ and $j \neq k$

$$\mathrm{E}\left[N_j(t)N_k(t) \mid X(0)=i\right] = \sum_{\ell=1}^{K} p_{i\ell} \int_0^t (h_{\ell jk}(t-t_1)+M_{\ell k}(t-t_1))dF_i(t_1) \tag{73}$$

The Laplace-Stieltjes transform of Equations 70 and 73 yields the desired result.

□

Proof for Theorem 2:

The proof involves the application of the law of total probability. Let T_1 be the time of the first transition. Let $t = 0$ denote the randomly selected point in time we begin to observe the process. Let $A_i(t) = \Pr\{T_1 \leq t \mid X(0) = i\}$, and let $\theta = \sum_{k=1}^{K} \pi_k \theta_k$. By Theorem 10.4.3 of Çinlar (1975b), ν_j is the probability that the process is in state j at a randomly selected point in time, where

$$\nu_j = \lim_{t\to\infty} P_{ij}(t) = \frac{\pi_j \theta_j}{\sum_{k=1}^{K} \pi_k \theta_k} \tag{74}$$

Thus,

$$\mathrm{E}\left[N_j(t)\right] = \sum_{i=1}^{K} \mathrm{E}\left[N_j(t) \mid X(0)=i\right] \Pr\{X(0)=i\}$$

$$
\begin{aligned}
&= \sum_{\substack{i=1\\ i\neq j}}^{K} \nu_i \sum_{k=1}^{K} p_{ik} \int_0^t M_{kj}(t-t_1)dA_i(t_1) + \\
&\quad \nu_j \sum_{k=1}^{K} p_{jk} \int_0^t (1 + M_{kj}(t-t_1)dA_j(t_1) \\
&= \nu_j A_j(t) + \sum_{i=1}^{K} \nu_i \sum_{k=1}^{K} p_{ik} \int_0^t M_{kj}(t-t_1)dA_i(t_1) \qquad (75)
\end{aligned}
$$

Taking the Laplace-Stieltjes transform yields

$$
\tilde{M}_j(s) \;=\; \nu_j \tilde{A}_j(s) + \sum_{i=1}^{K} \nu_i \tilde{A}_i(s) \sum_{k=1}^{K} p_{ik} \tilde{M}_{kj}(s) \qquad (76)
$$

Note that $A_i(t)$ is an equilibrium excess distribution and thus

$$
\tilde{A}_i(s) = \frac{1 - \tilde{F}_i(s)}{\theta_i s} \qquad (77)
$$

see Wolff (1989), pg. 66. To prove

$$
\tilde{M}_j(s) = \frac{\nu_j}{\theta_j s} \qquad (78)
$$

substitute Equation 77 into Equation 76, note that $\pi_k = \sum_{i=1}^{K} \pi_i p_{ik}$, and use Theorem 1. Equation 76 becomes

$$
\begin{aligned}
\tilde{M}_j(s) &= \frac{\pi_j(1-\tilde{F}_j(s))}{\theta s} + \sum_{i=1}^{K} \frac{\pi_i(1-\tilde{F}_i(s))}{\theta s} \sum_{k=1}^{K} p_{ik} \tilde{M}_{kj}(s) \\
&= \frac{\pi_j(1-\tilde{F}_j(s))}{\theta s} + \frac{1}{\theta s} \sum_{k=1}^{K} \pi_k \tilde{M}_{kj}(s) - \\
&\quad \frac{1}{\theta s}\left(\sum_{i=1}^{K} \pi_i \tilde{M}_{ij}(s) - \pi_j \tilde{M}_{jj}(s) + \pi_j \tilde{F}_j(s) \sum_{k=1}^{K} p_{jk} \tilde{M}_{kj}(s) \right) \\
&= \frac{\pi_j(1-\tilde{F}_j(s))}{\theta s} + \frac{\pi_j}{\theta s}\left(\tilde{F}_j + \tilde{F}_j(s) \sum_{k=1}^{K} p_{jk} \tilde{M}_{kj}(s) \right) - \\
&\quad \frac{\pi_j}{\theta s} \tilde{F}_j(s) \sum_{k=1}^{K} p_{jk} \tilde{M}_{kj}(s) \\
&= \frac{\pi_j}{\theta s}
\end{aligned}
$$

The results for $\mathrm{E}\left[N_j^2(t)\right]$ and $\mathrm{E}\left[N_j(t)N_k(t)\right]$ follow the similar arguments.

□

Proof for Theorem 3:

Let $\tilde{\mathbf{M}}(s) = [M_{ij}(s)]$, $\tilde{\mathbf{Q}}(s) = [p_{ij}F_i(s)]$, $\tilde{\mathbf{B}}(s) = \mathrm{diag}(F_1(s), \ldots, F_K(s))$, to show that

$$\tilde{\mathbf{M}}(s) = \left[\mathbf{I} - \tilde{\mathbf{Q}}(s)\right]^{-1} \tilde{\mathbf{B}}(s) \tag{79}$$

it is enough to show

$$\tilde{\mathbf{M}}(s) = \tilde{\mathbf{B}}(s) + \tilde{\mathbf{Q}}(s)\tilde{\mathbf{M}}(s) \tag{80}$$

Let $\tilde{\mathbf{C}}(s) = \tilde{\mathbf{Q}}(s)\tilde{\mathbf{M}}(s)$, then by the definition of matrix multiplication, we have

$$\tilde{C}_{ij}(s) = \tilde{F}_i(s) \sum_{k=1}^{K} p_{ik}\tilde{M}_{kj}(s) \tag{81}$$

Let $\tilde{\mathbf{D}}(s) = \tilde{\mathbf{B}}(s) + \tilde{\mathbf{Q}}(s)\tilde{\mathbf{M}}(s)$ thus

$$\tilde{D}_{ij}(s) = \begin{cases} \tilde{C}_{ij}(s) & \text{if } i \neq j \\ \tilde{F}_i(s) + \tilde{C}_{ii}(s) & \text{if } i = j \end{cases} \tag{82}$$

Thus, $\tilde{\mathbf{D}}(s) = \tilde{\mathbf{M}}(s)$ as was to be shown. To show

$$\tilde{\mathbf{M}}(s) = \left[\tilde{\mathbf{H}}(s) + \mathbf{I}\right] \tilde{\mathbf{B}}(s) \tag{83}$$

we note that from Çinlar (1975b), pp. 11-12 and Pyke (1961b) that

$$\tilde{\mathbf{H}}(s) = \left[\mathbf{I} - \tilde{\mathbf{Q}}(s)\right]^{-1} - \mathbf{I}$$

which implies

$$\tilde{\mathbf{H}}(s) + \mathbf{I} = \left[\mathbf{I} - \tilde{\mathbf{Q}}(s)\right]^{-1}$$

so that substitution into Equation 79 yields the desired result.

□

Proof for Theorem 4:

To prove Equation (11), write Equation (3) as a matrix:

$$\tilde{\mathbf{W}}(s) = \tilde{\mathbf{B}}(s) + \tilde{\mathbf{Q}}(s)\tilde{\mathbf{W}}(s) + 2\left[\tilde{\mathbf{Q}}(s)\tilde{\mathbf{M}}(s)\right]^d \tag{84}$$

Now, note that

$$\begin{aligned} \tilde{\mathbf{Q}}(s)\tilde{\mathbf{M}}(s) &= \tilde{\mathbf{M}}(s) - \tilde{\mathbf{B}}(s) \\ \left[\tilde{\mathbf{Q}}(s)\tilde{\mathbf{M}}(s)\right]^d &= \left[\tilde{\mathbf{M}}(s) - \tilde{\mathbf{B}}(s)\right]^d \\ \tilde{\mathbf{M}}(s)\tilde{\mathbf{B}}(s)^{-1} &= \left[\mathbf{I} - \tilde{\mathbf{Q}}(s)\right]^{-1} \end{aligned}$$

which implies that

$$\begin{aligned}\tilde{\mathbf{W}}(s) &= \tilde{\mathbf{B}}(s) + \tilde{\mathbf{Q}}(s)\tilde{\mathbf{W}}(s) + 2\left[\tilde{\mathbf{Q}}(s)\tilde{\mathbf{M}}(s)\right]^d \\ &= \tilde{\mathbf{M}}(s)\left[2\tilde{\mathbf{B}}(s)^{-1}(\tilde{\mathbf{M}}(s) - \tilde{\mathbf{B}}(s))^d + \mathbf{I}\right] \\ &= \tilde{\mathbf{M}}(s)\left[2(\tilde{\mathbf{M}}(s)\tilde{\mathbf{B}}^{-1}(s))^d - \mathbf{I}\right]\end{aligned}$$

since $\tilde{\mathbf{B}}(s)$ is diagonal. To show Equation (12), rewrite Equation (4) as

$$\tilde{h}_{i(jk)}(s) = \begin{cases} \sum_{\ell=1}^{K} \tilde{Q}_{i\ell}(s)\tilde{h}_{\ell(jk)}(s) & \text{if } i \neq j, i \neq k, j \neq k \\ \sum_{\ell=1}^{K} \tilde{Q}_{i\ell}(s)\tilde{h}_{\ell(jk)}(s) + \sum_{\ell=1}^{K} \tilde{Q}_{i\ell}(s)\tilde{M}_{\ell k}(s) & \text{if } i = j \text{ and } j \neq k \end{cases} \tag{85}$$

which yields

$$\tilde{h}_{i(jk)}(s) = \begin{cases} \sum_{\ell=1}^{K} \tilde{Q}_{i\ell}(s)\tilde{h}_{\ell(jk)}(s) & \text{if } i \neq j, i \neq k, j \neq k \\ \sum_{\ell=1}^{K} \tilde{Q}_{i\ell}(s)\tilde{h}_{\ell(jk)}(s) + \tilde{M}_{ik}(s) & \text{if } i = j \text{ and } j \neq k \end{cases} \tag{86}$$

Noting that

$$m_i = \begin{cases} 0 & \text{if } i \neq j \text{ and } i \neq k \\ M_{jk}(t) & \text{if } i = j \\ M_{kj}(t) & \text{if } i = k \end{cases}$$

gives

$$\vec{\tilde{\mathbf{h}}}^{(jk)}(s) = \tilde{\mathbf{Q}}(s)\vec{\tilde{\mathbf{h}}}^{(jk)}(s) + \vec{\tilde{\mathbf{m}}}^{(jk)}(s) \tag{87}$$

Substituting $\tilde{\mathbf{M}}(s)\tilde{\mathbf{B}}(s)^{-1} = \left[\mathbf{I} - \tilde{\mathbf{Q}}(s)\right]^{-1}$ yields the desired result.
□

Proof for Lemma 1:
Equations 13 and 14 are standard results. See for example Çinlar (1975b), pg. 329. To show Equations 15 and 16, we use Corollary 2.1.1 in Hunter (1969), pg. 192 which states that

$$m_{ij} = \sum_{k \neq j} p_{ik} m_{kj} + \theta_i \tag{88}$$

$$m_{ij}^{(2)} = \sum_{k \neq j} p_{ik} m_{kj}^{(2)} + 2\sum_{k \neq j} p_{ik}\theta_i m_{kj} + \theta_i^{(2)} \tag{89}$$

where $m_{ij}^{(r)}$ is the rth moment of the first passage time from state i to state j, and $\theta_i^{(r)}$ is the rth moment of the sojourn time distribution function. Multiplying both sides of Equation 89 by π_i and adding over all i yields

$$\begin{aligned}\sum_{i=1}^{K}\pi_i m_{ij}^{(2)} &= \sum_{i=1}^{K}\pi_i(\sum_{k=1}^{K}p_{ik}m_{kj}^{(2)} - p_{ij}m_{jj}^{(2)}) + \\ &\quad 2\sum_{i=1}^{K}\pi_i\theta_i(\sum_{k\neq j}p_{ik}m_{kj}) + \\ &\quad \sum_{i=1}^{K}\pi_i\theta_i^{(2)}\end{aligned}$$

Using Equation 88 yields,

$$\begin{aligned}\sum_{i=1}^{K}\pi_i m_{ij}^{(2)} &= \sum_{i=1}^{K}\pi_k m_{kj}^{(2)} - \pi_j m_{jj}^{(2)} + \\ &\quad 2\sum_{i=1}^{K}\pi_i\theta_i(m_{ij} - \theta_i) + \\ &\quad \sum_{i=1}^{K}\pi_i\theta_i^{(2)}\end{aligned}$$

simplifying yields,

$$\pi_j m_{jj}^{(2)} = \sum_{i=1}^{K}\pi_i\theta_i^{(2)} + 2\sum_{i=1}^{K}\pi_i\theta_i m_{ij} - 2\sum_{i=1}^{K}\pi_i\theta_i^2 \tag{90}$$

For two states,

$$\begin{array}{cc} m_{11} = (\pi_1\theta_1 + \pi_2\theta_2)/\pi_1 & m_{12} = \theta_1/(1-p_{11}) \\ m_{21} = \theta_2/(1-p_{22}) & m_{22} = (\pi_1\theta_1 + \pi_2\theta_2)/\pi_2 \end{array}$$

Note that $\alpha_i = m_{ii}$, $\sigma_i^2 = m_{ii}^{(2)} - (m_{ii})^2$, and $\delta_i^2 = \theta_i^{(2)} - (\theta_i)^2$ so that for state 1

$$m_{11}^{(2)} = \frac{\pi_1\delta_1^2 + \pi_2\delta_2^2}{\pi_1} + \frac{\pi_1\theta_1^2 + 2\pi_2\theta_1\theta_2 - \pi_2\theta_2^2}{\pi_1} + \frac{2\pi_2\theta_2^2}{\pi_1(1-p_{22})}$$

and thus

$$\sigma_1^2 = \frac{\pi_1\delta_1^2 + \pi_2\delta_2^2}{\pi_1} - \frac{\pi_2\theta_2^2}{\pi_1^2} + \frac{2\pi_2\theta_2^2}{\pi_1(1-p_{22})}$$

Noting that $p_{22} = 1 - \pi_1\beta$ yields

$$\sigma_1^2 = \frac{\pi_1\delta_1^2 + \pi_2\delta_2^2}{\pi_1} + \frac{\pi_2\theta_2^2}{\pi_1^2}\left(\frac{2}{\beta} - 1\right)$$

The proof for σ_2^2 is similar.

□

Proof for Lemma 2:

Because of the amount of algebraic manipulation involved in achieving the desired result, we will illustrate the necessary steps needed to obtain $\tilde{M}_{11}(s)$ while omitting the details and note that we utilized Maple (1991) as a check on the computations. From Theorem 1 the two state solution for $\tilde{M}_{11}(s)$ is

$$\tilde{M}_{11}(s) = \frac{\tilde{F}_1(s)(1 - p_{22}\tilde{F}_2(s))}{(1 - p_{11}\tilde{F}_1(s))(1 - p_{22}\tilde{F}_2(s)) - p_{12}p_{21}\tilde{F}_1(s)\tilde{F}_2(s))} \tag{91}$$

Expand $\tilde{F}_1(s)$ and $\tilde{F}_2(s)$ into Taylor series approximations as follows

$$\tilde{F}_1(s) = 1 - \theta_1 s + \theta_1^{(2)}\frac{s^2}{2} - \theta_1^{(3)}\frac{s^3}{6} + o(s^3) \tag{92}$$

$$\tilde{F}_2(s) = 1 - \theta_2 s + \theta_2^{(2)}\frac{s^2}{2} - \theta_2^{(3)}\frac{s^3}{6} + o(s^3) \tag{93}$$

where $\theta_i^{(r)}$ is the rth moment of the sojourn time distribution for state i. Substitution into Equation 91 expanding and collecting terms yields

$$\tilde{M}_{11}(s) = \frac{n_0 + n_1 s + n_2 s^2 + n_3 s^3 + o(s^3)}{d_1 s + d_2 s^2 + d_3 s^3 + o(s^3)} \tag{94}$$

where

$$\begin{aligned}
n_0 &= 1 - p_{22} \\
n_1 &= p_{22}\theta_2 - p_{21}\theta_1 \\
n_2 &= -\frac{p_{22}\theta_2^{(2)}}{2} - \theta_1 p_{22}\theta_2 + \frac{\theta_1^{(2)}p_{21}}{2} \\
n_3 &= \frac{p_{22}\theta_2^{(3)}}{6} + \frac{\theta_1 p_{22}\theta_2^{(2)}}{2} + \frac{\theta_1^{(2)}p_{22}\theta_2}{2} - \frac{\theta_1^{(3)}p_{21}}{6} \\
d_1 &= p_{12}\theta_2 + p_{21}\theta_1 \\
d_2 &= -\frac{p_{12}\theta_2^{(2)}}{2} - (1 - p_{11} - p_{22})\,\theta_1\theta_2 - \frac{\theta_1^{(2)}p_{21}}{2} \\
d_3 &= \frac{p_{12}\theta_2^{(3)}}{6} + \frac{(1 - p_{11} - p_{22})\,\theta_1\theta_2\theta_1^{(2)}\theta_2^{(2)}}{2} - \frac{\theta_1^{(3)}p_{21}}{6}
\end{aligned}$$

Expanding Equation 94 into a Taylor series about $s = 0$ yields

$$\begin{aligned}
\tilde{M}_{11}(s) &= \frac{n_0}{d_1 s} + \left(\frac{n_1 d_1 - n_0 d_2}{d_1^2}\right) + \\
&\quad \left(\frac{n_2 d_1^2 - n_0 d_3 d_1 - n_1 d_2 d_1 + n_0 d_2^2}{d_1^3}\right) s + o(s)
\end{aligned} \tag{95}$$

Substitution of n_0, n_1, n_2, n_3, d_1, d_2, d_3 into Equation 95 and using the relationships given in Lemma 1 yields the desired result. The solutions for $\tilde{M}_{12}(s)$, $\tilde{M}_{21}(s)$, $\tilde{M}_{22}(s)$ follow the same general pattern. The formulas for the c_{ij} are thus the functions of n_i and d_i given by the coefficient of the s in Equation 95.

□

Proof for Theorem 5:

Because of the amount of algebraic manipulation involved in achieving the desired result we will give the necessary steps needed while omitting the details and noting that we utilized Maple (1991) to check the computations. We drop the "(s)" on the Laplace-Stieltjes transform for notational ease. Expansion of the equations given in Theorem 2 for two states yields

$$\tilde{W}_1 = \frac{\tilde{Y}_1}{\theta s}\left[\tilde{X}_1\tilde{M}_{11} + \tilde{X}_2\tilde{M}_{21} + \pi_1(1-\tilde{F}_1)\right] \quad (96)$$

$$\tilde{W}_2 = \frac{\tilde{Y}_2}{\theta s}\left[\tilde{X}_1\tilde{M}_{12} + \tilde{X}_2\tilde{M}_{22} + \pi_2(1-\tilde{F}_2)\right] \quad (97)$$

$$\tilde{h}_{12} = \frac{\tilde{X}_1}{\theta s}\left[\tilde{Z}_1\tilde{M}_{11} + \tilde{Z}_2\tilde{M}_{12}\right] + \frac{\tilde{X}_2}{\theta s}\left[\tilde{Z}_1\tilde{M}_{21} + \tilde{Z}_2\tilde{M}_{22}\right] + \frac{\pi_1\tilde{Z}_1}{\theta s}(1-\tilde{F}_1) + \frac{\pi_2\tilde{Z}_2}{\theta s}(1-\tilde{F}_2) \quad (98)$$

$$\tilde{Y}_1 = 1 + 2\left[p_{11}\tilde{M}_{11} + p_{12}\tilde{M}_{21}\right] \quad (99)$$

$$\tilde{Y}_2 = 1 + 2\left[p_{21}\tilde{M}_{12} + p_{22}\tilde{M}_{22}\right] \quad (100)$$

$$\tilde{X}_1 = \pi_1 - \left[\pi_1 p_{11}\tilde{F}_1 + \pi_2 p_{21}\tilde{F}_2\right] \quad (101)$$

$$\tilde{X}_2 = \pi_2 - \left[\pi_1 p_{12}\tilde{F}_1 + \pi_2 p_{22}\tilde{F}_2\right] \quad (102)$$

$$\tilde{Z}_1 = p_{11}\tilde{M}_{12} + p_{12}\tilde{M}_{22} \quad (103)$$

$$\tilde{Z}_2 = p_{21}\tilde{M}_{11} + p_{22}\tilde{M}_{21} \quad (104)$$

Perform the following steps to achieve the desired results

1. substitute Taylor series expansions of $\tilde{F}_1$, $\tilde{F}_2$ into above equations
2. substitute expansions of $\tilde{M}_{ij}$ into above equations
3. collect coefficients of powers of s to form function with form

$$f(s) = \frac{k_2}{s^2} + \frac{k_1}{s} + k_0 + o(s)$$

4. invert functions to time domain

We now state B_1, D_1, and A_1 for completeness.

$$\begin{aligned}
B_1 \;=\;& \frac{\left(-2\,\pi_1^3\theta_1^{(3)}\theta_1 - 12\,\theta_2^3\theta_1\pi_1^2\pi_2 + 6\,\theta_1^{(2)}\;\pi_1^2\pi_2\theta_1\theta_2\right)\pi_1}{6\,\left(\pi_1\theta_1+\pi_2\theta_2\right)^4} + \\
& \frac{\left(12\,\theta_2^{(2)}\pi_1^2\pi_2\theta_1\theta_2 + 6\,\pi_1^2\pi_2\theta_2^{(2)}\theta_1^{(2)} - 6\,\pi_1^2\theta_1^2\theta_2^{(2)}\pi_2\right)\pi_1}{6\,\left(\pi_1\theta_1+\pi_2\theta_2\right)^4} + \\
& \frac{\left(12\,\theta_1^2\theta_2^2\pi_2^2\pi_1 + 6\,\pi_2^2\theta_2^{(2)}\pi_1\theta_1\theta_2 + 3\,\pi_1^3(\theta_1^{(2)})^2\right)\pi_1}{6\,\left(\pi_1\theta_1+\pi_2\theta_2\right)^4} + \\
& \frac{\left(12\,\theta_2^2\theta_1^2\pi_1^2\pi_2 - 12\,\theta_1\theta_2^3\pi_1\pi_2^2 + 3\,\pi_1\pi_2^2(\theta_2^{(2)})^2\right)\pi_1}{6\,\left(\pi_1\theta_1+\pi_2\theta_2\right)^4} + \\
& \frac{\left(-6\,\pi_2^2\pi_1\theta_1^{(2)}\theta_2^2 - 2\,\pi_1\pi_2^2\theta_2^{(3)}\theta_2 - 2\,\pi_1^2\pi_2\theta_2^{(3)}\theta_1\right)\pi_1}{6\,\left(\pi_1\theta_1+\pi_2\theta_2\right)^4} + \\
& \frac{\left(-12\,\pi_1^2\theta_1^{(2)}\theta_2^2\pi_2 - 2\,\pi_1^2\theta_1^{(3)}\pi_2\theta_2\right)\pi_1}{6\,\left(\pi_1\theta_1+\pi_2\theta_2\right)^4} + \\
& \frac{\left(24\,\theta_1\theta_2^3\pi_1\pi_2 - 12\,\theta_2^{(2)}\pi_2\pi_1\theta_1\theta_2 + 12\,\theta_1^{(2)}\theta_2^2\pi_1\pi_2\right)\pi_1}{6\,\left(\pi_1\theta_1+\pi_2\theta_2\right)^4\beta} + \\
& \frac{\left(12\,\theta_1\theta_2^3\pi_2^2 - 12\,\theta_1^2\theta_2^2\pi_1\pi_2\right)\pi_1}{6\,\left(\pi_1\theta_1+\pi_2\theta_2\right)^4\beta} - \\
& \frac{2\,\theta_1\theta_2^3\pi_1\pi_2}{\beta^2\left(\pi_1\theta_1+\pi_2\theta_2\right)^4}
\end{aligned} \tag{105}$$

$$\begin{aligned}
D_1 \;=\;& \frac{\left(-12\,\theta_1^3\theta_2\pi_2^2\pi_1 + 6\,\pi_2^2\theta_2^{(2)}\pi_1\theta_1^{(2)}\right)\pi_2}{6\,\left(\pi_1\theta_1+\pi_2\theta_2\right)^4} + \\
& \frac{\left(-12\,\pi_2^2\theta_1^2\theta_2^{(2)}\pi_1 - 6\,\pi_2^2\pi_1\theta_1^{(2)}\theta_2^2 - 2\,\pi_2^2\pi_1\theta_1^{(3)}\theta_2\right)\pi_2}{6\,\left(\pi_1\theta_1+\pi_2\theta_2\right)^4} + \\
& \frac{\left(-2\,\pi_2\pi_1^2\theta_1^{(3)}\theta_1 - 2\,\pi_2^3\theta_2^{(3)}\theta_2 + 12\,\theta_1^{(2)}\pi_2^2\pi_1\theta_1\theta_2 + 12\,\theta_1^2\theta_2^2\pi_2^2\pi_1\right)\pi_2}{6\,\left(\pi_1\theta_1+\pi_2\theta_2\right)^4} + \\
& \frac{\left(-6\,\pi_2^2\theta_2^{(2)}\pi_1\theta_1\theta_2 + 3\,\pi_2\pi_1^2(\theta_1^{(2)})^2 + 3\,\pi_2^3(\theta_2^{(2)})^2 - 2\,\pi_2^2\theta_2^{(3)}\pi_1\theta_1\right)\pi_2}{6\,\left(\pi_1\theta_1+\pi_2\theta_2\right)^4} + \\
& \frac{\left(-6\,\pi_1^2\theta_1^2\theta_2^{(2)}\pi_2 + 6\,\theta_1^{(2)}\pi_1^2\pi_2\theta_1\theta_2 + 12\,\theta_2^2\theta_1^2\pi_1^2\pi_2 - 12\,\theta_1^3\theta_2\pi_2\pi_1^2\right)\pi_2}{6\,\left(\pi_1\theta_1+\pi_2\theta_2\right)^4} -
\end{aligned}$$

$$
\frac{\left(-12\,\theta_1^2\theta_2^{(2)}\pi_2\pi_1 - 24\,\theta_1^3\theta_2\pi_2\pi_1 + 12\,\theta_1^{(2)}\theta_2\pi_2\pi_1\theta_1\right)\pi_2}{6\,\left(\pi_1\theta_1+\pi_2\theta_2\right)^4\beta} +
\frac{\left(12\,\theta_1^2\theta_2^2\pi_1\pi_2 - 12\,\pi_1^2\theta_2\theta_1^3\right)\pi_2}{6\,\left(\pi_1\theta_1+\pi_2\theta_2\right)^4\beta} -
\frac{2\,\theta_1^3\theta_2\pi_2\pi_1}{\beta^2\left(\pi_1\theta_1+\pi_2\theta_2\right)^4} \tag{106}
$$

$$
\begin{aligned}
A_1 \;=\; & \frac{\left(-6\,\theta_1\theta_2^3\pi_1\pi_2 - 6\,\theta_1^3\theta_2\pi_2\pi_1 + 6\,\pi_2\theta_2^{(2)}\pi_1\theta_1^{(2)}\right)\pi_2\pi_1}{6\,\left(\pi_1\theta_1+\pi_2\theta_2\right)^4} + \\
& \frac{\left(3\,\pi_2^2(\theta_2^{(2)})^2 + 3\,\pi_1^2(\theta_1^{(2)})^2 + 9\,\theta_1^{(2)}\theta_2\pi_2\pi_1\theta_1 - 2\,\pi_2^2\theta_2^{(3)}\theta_2\right)\pi_2\pi_1}{6\,\left(\pi_1\theta_1+\pi_2\theta_2\right)^4} + \\
& \frac{\left(-2\,\pi_2\theta_2^{(3)}\pi_1\theta_1 - 9\,\theta_1^2\theta_2^{(2)}\pi_2\pi_1 - 9\,\theta_1^{(2)}\theta_2^2\pi_1\pi_2\right)\pi_2\pi_1}{6\,\left(\pi_1\theta_1+\pi_2\theta_2\right)^4} + \\
& \frac{\left(-2\,\pi_1\theta_1^{(3)}\pi_2\theta_2 + 12\,\theta_1^2\theta_2^2\pi_1\pi_2 - 2\,\pi_1^2\theta_1^{(3)}\theta_1\right)\pi_2\pi_1}{6\,\left(\pi_1\theta_1+\pi_2\theta_2\right)^4} + \\
& \frac{\left(9\,\theta_2^{(2)}\pi_2\pi_1\theta_1\theta_2 + 6\,\pi_1^2\theta_1^2\theta_2^2 - 6\,\theta_1\theta_2^3\pi_2^2\right)\pi_2\pi_1}{6\,\left(\pi_1\theta_1+\pi_2\theta_2\right)^4} + \\
& \frac{\left(3\,\theta_1\pi_2^2\theta_2\theta_2^{(2)} + 6\,\theta_1^2\pi_2^2\theta_2^2 - 3\,\pi_2^2\theta_2^2\theta_1^{(2)}\right)\pi_2\pi_1}{6\,\left(\pi_1\theta_1+\pi_2\theta_2\right)^4} + \\
& \frac{\left(-6\,\pi_1^2\theta_2\theta_1^3 - 3\,\pi_1^2\theta_1^2\theta_2^{(2)} + 3\,\pi_1^2\theta_1\theta_2\theta_1^{(2)}\right)\pi_2\pi_1}{6\,\left(\pi_1\theta_1+\pi_2\theta_2\right)^4} + \\
& \frac{\left(-6\,\theta_1^{(2)}\theta_2\pi_1\theta_1 + 6\,\theta_1\theta_2^3\pi_2 - 18\,\theta_1^2\theta_2^2\pi_1\right)\pi_2\pi_1}{6\,\left(\pi_1\theta_1+\pi_2\theta_2\right)^4\beta} + \\
& \frac{\left(-18\,\theta_1^2\theta_2^2\pi_2 + 6\,\theta_1^3\theta_2\pi_1 - 6\,\theta_1\theta_2^{(2)}\pi_2\theta_2\right)\pi_2\pi_1}{6\,\left(\pi_1\theta_1+\pi_2\theta_2\right)^4\beta} + \\
& \frac{\left(6\,\theta_1^2\theta_2^{(2)}\pi_1 + 6\,\theta_1^{(2)}\theta_2^2\pi_2\right)\pi_2\pi_1}{6\,\left(\pi_1\theta_1+\pi_2\theta_2\right)^4\beta} + \\
& \frac{2\,\theta_1^2\theta_2^2\pi_1\pi_2}{\beta^2\left(\pi_1\theta_1+\pi_2\theta_2\right)^4}
\end{aligned} \tag{107}
$$

This completes the proof.

□

Proof for Corollary 1:
To show the result, substitute

$$\tilde{F}_1 = \frac{\frac{1}{\theta_1}}{s + \frac{1}{\theta_1}}$$

$$\tilde{F}_2 = \frac{\frac{1}{\theta_2}}{s + \frac{1}{\theta_2}}$$

into the two state equations for W_1, W_2, and h_{12} given in the proof of Theorem 5 to get the following set of equations

$$\tilde{W}_1(s) = \frac{\left(s^2\theta_1\theta_2 + \left((\pi_1\theta_1 - \pi_2\theta_2)\,\beta + 2\,\theta_2\right)s + 2\,\pi_1\beta\right)\pi_1}{s^2\,(\beta\,\theta + s\theta_1\theta_2)\,\theta} \quad (108)$$

$$\tilde{W}_2(s) = -\frac{\left(\left((\pi_1\theta_1 - \pi_2\theta_2)\,\beta - 2\,\theta_1\right)s - s^2\theta_1\theta_2 - 2\,\pi_2\beta\right)\pi_2}{s^2\,(\beta\,\theta + s\theta_1\theta_2)\,\theta} \quad (109)$$

$$\tilde{h}_{12}(s) = \frac{((\theta_2 + \theta_1)\,s + 2)\,\pi_2\beta\,\pi_1}{s^2\,(\beta\,\theta + s\theta_1\theta_2)\,\theta} \quad (110)$$

Partial fraction expansion, substitution of the results of Lemma 1, and inversion yields the desired result.
□

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