

Evaluating the Lead Time Demand Distribution for (r, Q) Policies Under Intermittent Demand

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Abstract

This paper examines the use of standard (r, Q) inventory control policies for situations involving intermittent highly variable demand that is known to have sporadic or infrequent occurrences along with high variability for the number of units demanded. The paper investigates how well the inventory control model behaves under various demand scenarios. The key to the application of inventory models to intermittent scenarios is how well the lead-time demand distribution can be modeled. A number of distributions, namely, normal, gamma, Poisson and negative binomial that might be applicable in the case of intermittent and highly variable demand are examined. Historical data from industry was analyzed and used to develop realistic demand scenarios for the purpose of experimentation. Based on empirically estimated parameters, the experimental results compare performance measures; namely, fill rate and average backorder levels which are generated under each inventory control policy with a corresponding lead time demand distribution. The evaluation of the accuracy of these measures is also compared via a simulation study.

Keywords:

Intermittent demand, lead time demand distribution, inventory control policy

1. Introduction

Inventory managers set inventory policies with the objective of optimizing the operational performance measures of the inventory system. The calculation of operational performance at each policy level requires the modeling of the lead-time demand distribution or its components (lead time (LT) distribution and the demand (D) distribution). The modeling of the lead-time demand distribution is often performed by assuming the family of the distribution. Regardless of the assumed distributional family, estimates of the parameters of the distribution based on forecast estimates and the forecast error (or other knowledge) are used. Then, a particular inventory policy is used and the so-called "optimal" inventory policy parameters are calculated.

Unfortunately, the actual distribution of lead-time demand is difficult to directly measure in practice. In selecting the distribution, Adan et al. [1] and Axsäter [2] proposed different rules of thumb which can be used to select a lead-time demand distribution. The plethora of possible distributions leads to the question of which one is best for which conditions. Arbitrarily selected distribution based on an assumption or these rules of thumb built on estimated mean and variances can only approximately model the lead-time demand distribution. Based on the first two moments, the parameters of the distribution are computed. However, the required mean and variances are estimated and hence this introduces another layer of error into the lead-time demand modeling. That is, there is the error associated with arbitrarily picking a lead-time demand distribution family and the error associated with fitting its parameters. There is also the error of using the best inventory control model for the given situation.

These representational errors cause the planned for operational performance to not be met. This is a problem in the use of inventory models in general, but it is especially so in the case of intermittent demand where the characterization of the lead-time demand distribution is even more problematic. Demand for products that have sporadic or infrequent occurrences along with high variability for the number of units demanded is often termed intermittent demand. Intermittent demand data is also characterized by time series that have many time periods with zero demand. Intermittent demand often occurs for slower moving inventory items and for the demand processes resulting from repair operations. Various approaches have been developed in the literature to cope with the inventory situations involving intermittent demand [3-5]. Inventory control under intermittent demand is a challenging task, which is mostly due to the nature of the demand pattern. The transaction (demand incidence) variability (sporadicity) and also demand size variability (lumpiness) are two factors that together make it hard to model the lead time demand distribution, and accordingly set an appropriate inventory control policy. Due to these

reasons, researchers often tend to assume a distribution family that best represents the corresponding lead-time demand pattern. This paper focuses on the question of which lead-time demand distribution(s) are reasonable for which conditions.

2. Selecting the Family of Lead Time Demand Distribution

In the application of traditional inventory models, it is often assumed that the lead-time demand distribution is normal (Silver et al. [6]; Axsäter [2]). Syntetos and Boylan [4] state that the normal distribution may be utilized in inventory control policies as long as the demand distribution exhibits a small coefficient of variation (CV). They also claim that there is not enough support delivered by the literature on the performance of the normality assumption for the case of intermittent demand. Demand is assumed to follow a gamma process in Segerstedt [3] and Janssen et al. [7]. The latter authors argue against the normal distribution's popularity in inventory control management by asserting several reasons. First of all, the normal distribution does not fit the three criteria discussed in Burgin [8]. The gamma distribution is nonnegative and the value of the shape parameter can be adjusted to get desired forms. Larsen et al. [9] point out that compound Poisson demand processes are suggested in the inventory literature to model spare part items with lumpy and low frequency demand. Syntetos and Boylan [4] consider the negative binomial distribution as an appropriate lead time demand distribution since it satisfies the 3 criteria proposed by Boylan [10]: i) A priori grounds for modeling demand, ii) The flexibility of the distribution to represent different types of demand, iii) Empirical evidence.

The fact that the above mentioned distributions are often implemented or suggested by the literature in the realm of inventory control research is important since they are often implemented in practical software solutions. Thus, this research examines the use of standard (r, Q) inventory control policies under a number of lead-time demand distributions, namely, normal (N), gamma (G), Poisson (P) and negative binomial (NB) that might be applicable within the context of intermittent and highly variable demand. For simplicity, we assume a continuous review (r, Q) inventory control policy with back ordering permitted and deterministic lead times. This paper reports on initial results, with future research aimed at results for other policies under more general assumptions.

3. Analysis and Results

Inventory policies are typically determined in order to optimize (or satisfy) desired operational performance measures. In evaluating these performance measures based on the use of different distributions for a (r, Q) inventory model, one basic question arises under the assumption that the lead time demand distribution () is known with certainty: What would be the effect on inventory performance measures if the lead time demand distribution is assumed to be some other distribution ()? Clearly, aforementioned representational errors are of interest in order to identify the magnitude of the effect. In this respect, we define the variables and | | as the error and absolute relative error for the test case j. Let be the value of a performance measure for the test case j when the LTD is actually and let be the value of a performance measure for the test case j under the assumption that the LTD is . Thus,

$$= - \tag{1}$$

$$| | = \frac{ }{ } \tag{2}$$

In this paper, two operational performances are considered: fill rate and the expected number of backorders. Under the assumption that is the lead-time demand distribution, and are the performance measures of fill rate and expected number of backorders, respectively. Note that can be either or . The following general formulations (1) and (2)) are used in order to evaluate the desired service levels:

$$= 1 - [() - (+)] \tag{3}$$

$$= - [() - (+)] \tag{4}$$

where r and Q are the re-order point and the order quantity while and are the first and second order loss functions of F. As can be seen from the formulations, a performance measure is calculated for each policy parameters (r and Q) for the distribution . For performance measure calculations, the formulations presented in Zipkin [11] are used. The reader is referred to Zipkin [11] for details and further explanation of the formulations.

Axsäter [2] proposed a rule of thumb, which we call "Axsäter's Rule (AR)," in selecting the distribution to fit for lead-time demand. Demand with estimated coefficient of variation (CV) that lies within the range 0.9 and 1.1 can be fit with Poisson distribution. If the estimated CV is less than 0.9, then Poisson or a mixture of binomial (described in

Adan et al. [1]) can be considered; and if the estimated CV is greater than 1.1, the negative binomial can be selected. Note that the rule suggests using one of two distributions for those CV values less than 0.9. We drop the option of using the mixture of binomial distribution and, instead, use Poisson distribution for those CV values less than 1.1. The representational errors are calculated upon the rule selecting a distribution with respect to the mean and standard deviation of the demand during lead-time.

In Table-1, statistical error results of the service level measures of fill rate and the average number of backorders are given. Four types of statistics are of interest in our experiments: The average (\bar{E}), standard deviation (σ_E), median ($Med(E)$) of the error (E) and the probability that the absolute relative error ($|RE|$) is less than or equal to 0.10 ($P(0.10)$). These error statistics are recorded for each μ and σ with respect to an assumed distribution (\mathcal{D}) or a rule (Normal (N), Gamma (G), Negative Binomial (NB) or Axsäter's Rule (AR)), when the LTD is in fact known to follow a distribution (\mathcal{D} is one of N, P, G, NB). Statistics of these error results are given for 1000 different cases. The cases were generated with different mean (μ), standard deviation (σ) of lead-time demand and r, Q policy parameter values. Specifically, both μ and σ are sampled from a uniform distribution over the range (0, 10] and (0.7, 5.5), respectively while Q and r are sampled from a discrete uniform distribution over the range [1, 100] and [- Q , +2], respectively. This empirical data scheme mostly represents real intermittent demand data obtained from industrial organizations collaborating with this research effort. However, the empirical data scheme is not applied to the feasible regions of all distributions. Thus, not all generated cases will be over the same ranges for μ and σ for every assumed distribution. For example, if \mathcal{D} is Poisson, then CV is not required to be sampled, since the mean and the variance of the LTD are the same. The following algorithm summarizes the recording of the error statistics.

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Generate test cases of (Q, r)
For  $\mathcal{D}$  in {P, NB, N, G}, do
    Assume a range of  $E[X]$  and CV that is feasible for  $\mathcal{D}$  and (if possible)
    representative of intermittent data
    Match moments for  $\mathcal{D}$  in {P, NB, N, G, AR}
    Evaluate  $\bar{E}$  and  $\sigma_E$  for  $\mathcal{D}$  and  $\mathcal{R}$  using (3) and (4)
    Evaluate  $Med(E)$  and  $P(|RE| \leq 0.10)$  for each test case  $j$  using (1) and (2)
    Record the error statistics
End do
    
```

Figure-1: Error statistics are recorded based on the given algorithm

The method of matching moments between distributions is given as follows: Let μ and σ be the mean and standard deviation of demand during lead time. We represent a normal distributed lead time demand variable $X \sim \mathcal{N}(\mu, \sigma)$. The mean and variance of the normal distribution are $E[X] = \mu$ and $V[X] = \sigma^2$. The parameters of other distributions are as follows: Gamma distribution has two parameters α (shape) and β (scale). We represent gamma distributed random lead time demand variables as $X \sim \mathcal{G}(\alpha, \beta)$ where ($\alpha \geq 0$). The mean and variance of the gamma distribution $E[X] = \alpha\beta$, $V[X] = \alpha\beta^2$ and parameters are fit by: $\alpha = \frac{\mu^2}{\sigma^2}$ and $\beta = \frac{\sigma^2}{\mu}$. Poisson distribution has one parameter $\lambda > 0$ We represent Poisson distributed random lead time demand variables as $X \sim \mathcal{P}(\lambda)$ where ($\lambda \in \mathbb{Z}$). The mean and variance of the Poisson distribution $E[X] = V[X] = \lambda$, and its parameter is fit by $\lambda = \mu$. Negative Binomial distribution has two parameters, $n > 0$ and $0 < p < 1$ where n is number of 0's before 1's occur in a Bernoulli sequence and p is the probability of having 1 in each trial. We represent negative binomial distributed lead time demand variables as $X \sim \mathcal{NB}(n, p)$ where ($n \in \mathbb{Z}$). The mean and variance of the negative binomial distribution $E[X] = \frac{np}{1-p}$ and $V[X] = \frac{np}{(1-p)^2}$. The parameters are fit by $p = \frac{\sigma^2}{\mu^2}$ and $n = \frac{\mu^2}{\sigma^2 - \mu}$. Note that n is not defined if $\sigma^2 = \mu$. In this case, we set $n = 1$ and then $p = \frac{\mu}{\sigma^2}$.

Table 1 presents the comparisons for each of the assumed distributions. On the left side of the table (the first columns), the assumed correct distribution is given. Across the body of the table is the distribution used to approximate the true situation. As indicated, whenever the approximating distribution matches the assumed distribution, there is no error. In almost all the cases, the error values associated with the expected number of

Table 1: Fill rate () and the average number of backorder () error results (, , and P(0.1))

		Normal (N)				Gamma (G)				Poisson (P)				Negative Binomial (NB)				Axsäter's Rule (AR)			
					P(.1)(%)				P(.1)(%)				P(.1)(%)				P(.1)(%)				P(.1)(%)
N		-	-	-	-	-0.0241	0.0764	-0.0030	76.50	0.0158	0.0946	0.0063	67.00	-0.0243	0.0766	-0.0041	75.30	-0.0240	0.0772	-0.0041	74.50
		-	-	-	-	0.2924	1.8733	0.0009	42.10	-1.7297	2.3657	-0.7385	23.60	0.4572	1.9528	0.1306	37.00	0.4701	1.9529	0.1486	36.10
G		0.0241	0.0764	0.0030	76.50	-	-	-	-	-0.0084	0.0685	0.0013	76.60	-0.0002	0.0080	0.0000	94.80	0.0001	0.0153	0.0001	93.10
		-0.2924	1.8733	-0.0009	41.40	-	-	-	-	-1.4373	1.3100	-1.0044	30.60	0.1648	0.2207	0.0498	91.70	0.1777	0.2371	0.0526	90.30
P		0.0002	0.0094	0.0000	90.95	0.0001	0.0087	0.0000	90.75	-	-	-	-	-0.0011	0.0309	0.0000	85.33	0.0000	0.0002	0.0000	0.30
		-0.1726	0.2433	-0.0450	59.10	-0.1722	0.2416	-0.0464	50.20	-	-	-	-	-0.1739	0.2664	-0.0605	45.50	-0.0002	0.0013	0.0000	99.20
NB		0.0243	0.0766	0.0041	76.18	0.0002	0.0080	0.0000	95.18	-0.0085	0.0687	0.0000	76.28	-	-	-	-	0.0003	0.0131	0.0000	97.69
		-0.4572	1.9528	-0.1306	35.80	-0.1648	0.2207	-0.0498	90.50	-1.2725	1.3693	-0.7664	35.90	-	-	-	-	0.0129	0.0807	0.0000	97.70

Table 2: Experimental factors for the simulation model

Level	Q	r	LT			
Low	1	-1	0.5	2	2	2
High	4	4	1	4	4	4

Table 3: Fill rate () and the average number of backorder () error results based on the simulation model

	Fill rate ()					Average number of backorders ()				
	N	G	P	NB	AR	N	G	P	NB	AR
	-0.1937	-0.1583	-0.1618	-0.1578	-0.1581	0.0900	0.1377	0.4448	0.3301	0.3301
	0.0977	0.1477	0.1592	0.1461	0.1460	0.3222	0.3188	0.4356	0.3923	0.3923
Min.	-0.4116	-0.4897	-0.5750	-0.4828	-0.4828	-0.4127	-0.3190	0.0052	0.0008	0.0008
	-0.2531	-0.2895	-0.2362	-0.2914	-0.2914	-0.1421	-0.0385	0.0957	0.0444	0.0444
	-0.1900	-0.1281	-0.1265	-0.1307	-0.1307	0.0215	0.0346	0.2883	0.1933	0.1933
	-0.1307	-0.0106	-0.0144	-0.0114	-0.0114	0.2599	0.2199	0.6972	0.4291	0.4291
Max.	-0.0192	0.0000	0.0000	0.0000	0.0000	0.9353	1.0554	1.5554	1.5554	1.5554
P(.1)(%)	15.63	43.75	43.75	43.75	43.75	15.63	1.56	0.00	7.81	7.81

backorders are higher than those of fill rates. In the case where the actual distribution is normal, the best performance results are produced if the gamma distribution is used to approximate performance. In the case where the actual lead-time demand distribution is known to be gamma, the performance measure of fill rate is best approximated by both the negative binomial distribution and Axsäter's rule, giving the smallest range of error measures and most of the absolute relative error measures below 0.1. The performance measure results of fill rate and the average number of backorders are best estimated by the normal and gamma distributions if the lead-time demand distribution is known to be Poisson. In such cases, the performance measure of the expected number of backorders is best approximated by the Axsäter's rule. In the case where the actual lead-time demand follows a negative binomial distribution, the best performance measure results for both fill rate and the expected number of backorders are produced if the gamma distribution or Axsäter's rule is used.

In the second part of the analysis, the representational errors are determined using the results of a simulation study. The simulation model of a standard (r, Q) inventory system was built on a JAVA platform using the Java Simulation Library (JSL) [12]. The JSL is an open source simulation library developed for discrete event simulation modeling by supporting random number generation, statistical collection and basic reporting.

The simulation model consists of 3 modules. The first module sets the parameters for 64 different cases. These parameters are the combinations of order quantity (Q), re-order point (r), lead time (LT) and intermittent demand generator parameters (λ , α , β) given in Table-2. The second module utilizes an intermittent demand generator to draw the actual intermittent demand observations from the specified demand distributions. The most significant part of the simulation model is that it generates the intermittent demand by employing a flexible discrete-time arrival process, the so-called "batch on/off process" [13]. A group or batch of arrivals may be generated at any active slot, which, in fact, introduces a new approach in regard to three input random variables characterizing the process. These three input random variables characterize 1) busy period and 2) idle period and 3) batch size. Consequently, three general distributions are assumed for the above mentioned three input random variables. An on/off model consists of two alternating busy (on) and one idle (off) periods. Arrivals are generated during busy periods, while no arrivals are generated during idle periods. Successive busy and idle periods are independent and identically distributed. An on/off model can be regarded as the combination of two alternating independent renewal process, since busy and idle periods are also assumed to be independently generated from each other. What follows is the generation of a batch or group of arrivals at each active slot. We assume that each of batch sizes (B), number of busy periods (N) and number of idle periods (I) follows a geometric distribution with the success probability of $1/\lambda$, $1/\alpha$ and $1/\beta$, respectively. In the third module, the forecasting procedure is simulated through the revised Croston's forecasting technique [14] in order to estimate the mean and the variance of the lead time demand. The motivation behind using the revised Croston's forecasting method lies in the fact that it is often recommended by the literature and also considered by some innovative forecasting approaches given in [15].

The simulation model for each generated case was run for one replication with 360 time units warm-up and 672000 time units of run-time. The performance measures of fill rate and the average number of backorders are monitored during the simulation. For each performance measure, the maximum standard error of 0.004 is observed, which gives, in principle, 2 digits of accuracy in both the fill rate and the expected number of backorders.

The errors statistics given in Table-3 are evaluated based on the difference between the values of the measures gained by the simulation runs and those gained by a selected distribution (N, G, P, NB) or the rule (AR) of which parameters are fed by the forecasted mean and variance of the lead-time demand. Note that the first and the third quartiles (Q_1 and Q_3) are also reported in the table. The results associated with the performance measure of the fill rate yield negative error on average for each of N, G, NB and AR. In fact, each of distributions and the rule overestimate the fill rate, since the maximum recorded errors have negative or zero values at best. This is not considered a good result within an inventory planning context due to the fact that the inventory manager will be inclined to think that better fill rate can be achieved than the realistic rates. On the other hand, the average number of backorders is underestimated by the distributions and the rule. In this case, the manager will think that less backorders are achieved than the realistic levels. The normal distribution performs relatively well when approximating the expected number of backorders as it yields the smallest errors on average in the experiments. However, the worst performance when approximating the expected number of backorders is produced by the gamma distribution as it yields a huge variation for the observed errors.

4. Conclusion and Future Research

This paper examined the (r, Q) inventory policy in the context of intermittent demand with deterministic lead time. In the first part of the analysis, we have assumed that the demand during lead-time truly is known and follows one of the distributions of normal, gamma, Poisson or negative binomial. The presented error evaluation measures the effect on inventory performance metrics of fill rate and average backorder levels, if the inventory manager selects N, G, P, NB or AR to modeling lead-time demand. In the second part of the analysis, a simulation based analysis examined which distribution or rule better approximates the performance measures under the given experimental conditions.

Clearly, these initial results may not be enough to draw full conclusion. One direction for further research is to carry out a more comprehensive simulation analysis over a wider range of levels. Another direction lies in investigating different switching rules such as Adan et al. [1] or developing a new rule taking into account the results in this study. The last, but not least, direction is to develop a better distributional model for the lead time demand under intermittent demand conditions.

References

1. Adan I., van Eenige M., Resign J., 1995, "Fitting Discrete Distributions on the First Two Moments," *Probability in the Engineering and Informational Sciences*, 9, 623-632
2. Axsäter S., 2006, *Inventory control*, 2nd ed. Springer, New York.
3. Segerstedt A., 1994, "Inventory control with variation in lead times, especially when demand is intermittent," *International Journal of Production Economics*, 35, 365-72
4. Syntetos A. A., Boylan J. E., 2006, "On the stock control performance of intermittent demand estimators," *International Journal of Production Economics*, 103(1), 36-47
5. Dunsmuir W. T. M., Snyder R. D., 1989, "Control of inventories with intermittent demand," *European Journal of Operational Research*, 40(1), 16-21
6. Silver, E.A., Pyke D.F., Peterson R., 1998, "Inventory management and production planning and scheduling," 3rd ed. Wiley, New York
7. Janssen E., Strijbosch L. W. G., Brekelmans R. C. M., 2007, "How to Determine the Order-Up-To Level when Demand is Gamma Distributed with Unknown Parameters," *CentER Discussion Paper No. 2007-71*
8. Burgin T. A., 1975, "The Gamma Distribution and Inventory Control," *Operational Research Quarterly*, 26(3), 507-525
9. Larsen C., Seiding C. H., Teller C., Thorstenson A., 2008, "An inventory control project in a major Danish company using compound renewal demand models," *IMA Journal of Management Mathematics*, 19(2), 145-162
10. Boylan J.E., 1997, "The centralisation of inventory and the modelling of demand," Unpublished Ph.D. Thesis, university of Warwick, UK.
11. Zipkin P.H., 2000, "Foundations of inventory management," McGraw-Hill, Boston.
12. Rossetti M. D., Miman M., Varghese V., 2008, "An object-oriented framework for simulating supply systems," *Journal of Simulation*, 0, 1-14
13. Galmes S., Puigjaner R., 2003, "Correlation analysis of a discrete-time flexible arrival process," *Computer Networks*, 41(6), 795-814
14. Syntetos A. A. and Boylan J. E., 2001, "On the bias of intermittent demand estimates," *Int. J. Production Economics*, 71, 457-466
15. Varghese V., Rossetti M., 2009, "A meta forecasting methodology for large scale inventory systems with intermittent demand," submitted to *Proceedings of the 2009 Industrial Engineering Research Conference*