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Evaluating Performance Measure Approximations for a Two-Echelon (*R*, *Q*) Inventory System via Simulation

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Abstract

This study examines the quality of the solutions obtained via an analytical inventory model against a simulation model. The analytical inventory model is dependent upon approximate expressions for performance measures such as the average number of backorders. In this study, a one-warehouse, two-echelon inventory simulation model is developed and used to predict the performance measures for a given set of inventory policy parameters. The policy parameters are estimated using the optimization algorithm and used as an input parameters for the simulation model. The data shows that there is significant difference between the actual and analytical performance measures and suggests future areas of research.

Keywords
Inventory Model, Multi-Echelon, Performance Measures, Simulation.

1. Introduction
Multi-echelon inventory systems have been developed and widely used for managing spare part supply networks. Due to the complexity of modeling multi-echelon inventory systems approximate inventory performance measures have been developed. The quality of the solutions obtained using the optimization algorithms depends heavily on the accuracy of the approximate performance measures.

The main focus of this study is to examine the robustness of the inventory model presented by Al-Rifai and Rossetti [2] for the non-identical retailer case via simulation. The simulation model will allow us to better understand the behavior of the inventory model under the effect of different parameter values and levels and compare it to the solutions obtained using the analytical inventory model. Since the development of the original analytical model was based on a set of assumptions these models are not expected to capture the performance measures exactly, as captured by a simulation model. The simulation model is expected to result in more accurate performance measures. However, this study should provide initial insights for future research that might consider an extensive simulation study that could result in correction factors for the analytical inventory models.

The rest of this paper is organized as follows: In Section 2, we provide a review of the relevant literature. In Section 3, we discuss the analytical and simulation models. In Section 4, we experiment and analyze the results. In Section 5, we discuss the results. Finally, we conclude our investigation in Section 6.

2. Literature Review
Due to the complexity of modeling the non-identical retailer case much research has focused on the identical retailer case. Deuermeyer and Schwarz [4] developed a inventory model for a two-echelon inventory system that consists of one warehouse and N identical retailers that implements (R, Q) policies. Since they assumed Poisson demands and a (R, Q) policy is implemented, the demand process at the warehouse is a superposition of the retailer's ordering

processes. Since the demand rate at each retailer for each item is λ and the retailer's replenishment batch size is *Q*, the demand process at the warehouse is a superposition of renewal processes with *Q* stages and rate λ (Deuermeyer and Schwarz [4]). Due to the complexity of modeling the warehouse lead-time demand they approximated the demand process at the warehouse by a renewal process and derived approximate expressions for the warehouse's first two moments of lead-time demand. In another attempt for modeling the warehouse's lead-time demand for the identical retailer case Svoronos and Zipkin [6] proposed a refinement of the Deuermeyer and Schwarz [4] model. Svoronos and Zipkin [6] approximated the warehouse lead-time demand using a mixture of two translated Poisson distributions (MTP).

Al-Rifai and Rossetti [1] developed an optimization algorithm for a two-echelon inventory system with one warehouse and *N* identical retailers. In modeling the inventory performance measures they approximated the warehouse's and the retailer's lead-time demand using a normal distribution and used the expressions developed by Svoronos and Zipkin [6] for modeling the warehouse's first two moments of lead-time demand. On the other hand, they used the normal approximation for the Poisson to approximate the normal distribution parameters while modeling the retailer's lead-time demand. For the non-identical retailer case, Al-Rifai and Rossetti [2] extended the first two moments of the warehouse's lead-time demand developed by Svoronos and Zipkin [6] for the identical retailer case. In approximating the warehouse's first two moments of lead-time demand they used the retailer's average batch size instead of using a fixed batch size across the retailers. They also modeled the lead-time demand at the warehouse and at the retailer using a normal distribution.

The main focus of this study is to examine the robustness of the inventory model presented by Al-Rifai and Rossetti [2] for the non-identical retailer case via simulation. The inventory model under investigation consists of one warehouse supplying *N* non-identical independent retailers facing Poisson demands similar to the one presented by Al-Rifai and Rossetti [2]. No lateral shipments between the retailers are considered. A limited supply at the warehouse is considered and the warehouse is assumed to be supplied by a supplier with ample supply. A (*R*, *Q*) stocking policy is considered, so that when the inventory level at any facility drops below the reorder point (*R*), a replenishment order (*Q*) is placed and unsatisfied demands are backordered. Also, since the non-identical retailer case is considered the lead-times, demand rate, replenishment batch size, and the reorder point are allowed to vary across the retailers.

3. Analytical and Simulation Models

The complete analytical inventory model is presented by Al-Rifai and Rossetti (2007) which we repeat here for convenience. Under an (*R*, *Q*) policy the average on hand inventory at the retailer is given as:

$$\bar{I}_{ri}(R_{ri}, Q_{ri}) = \bar{B}_{ri}(R_{ri}, Q_{ri}) + R_{ri} + \frac{Q_{ri}+1}{2} - E[D_{ri}] \tag{1}$$

The retailer's expected lead time demand, $E[D_{ri}]$ and lead time, ℓ_{ri} are given as follows:

$$E[D_{ri}] = \lambda_{ri} \times \ell_{ri} \tag{2}$$

$$\ell_{ri} = L_{ri} + d_{ri} \tag{3}$$

$$d_{ri} = \bar{B}_{wi}(R_{wi}, Q_{wi}) / \lambda_{wi} \tag{4}$$

Where d_{ri} is the delay at the warehouse due to stockout. The demand rate at the warehouse, λ_{wi} and the retailer's effective batch size, Q_i^{eff} are given as follows:

$$\lambda_{wi} = \sum_{r=1}^{m} \frac{\lambda_{ri}}{Q_i^{eff}} = \frac{1}{Q_i^{eff}} \sum_{r=1}^{m} \lambda_{ri} \tag{5}$$

$$Q_i^{eff} = \frac{1}{m} \sum_{r=1}^{m} Q_{ri} \tag{6}$$

The mean and variance of the warehouse's lead time demand are given as follows:

$$E[D_{wi}] = \frac{L_{wi}}{Q_i^{eff}} \sum_{r=1}^{m} \lambda_{ri} \tag{7}$$

$$V[D_{wi}] = \frac{m^2}{\left(\sum_{r=1}^{m} Q_{ri}\right)^2}\left(L_{wi}\sum_{r=1}^{m}\lambda_{ri} + \sum_{r=1}^{m}\sum_{k=1}^{a_0}\left(\frac{[1-\exp(-\alpha_k\lambda_{ri}L_{wi})\cos(\beta_k\lambda_{ri}L_{wi})]}{\alpha_k}\right)\right) \tag{8}$$

Where, $$\alpha_k = 1-\cos(2\pi k / Q_i^{eff}) \text{ and } \beta_k = \sin(2\pi k / Q_i^{eff}) \tag{9}$$

$$a_0 = \begin{cases} Integer(Q_i^{eff} - 1), & if\ Q_i^{eff} - 1 \geq 1 \\ 1 & , otherwise \end{cases} \tag{10}$$

Item i's average on-hand inventory at the warehouse is:

$$\bar{I}_{wi}(R_{wi}, Q_{wi}) = \bar{B}_{wi}(R_{wi}, Q_{wi}) + R_{wi} + \frac{Q_{wi}+1}{2} - \frac{L_{wi}}{Q_i^{eff}}\sum_{r=1}^{m}\lambda_{ri} \tag{11}$$

Under a (R, Q) policy, item i's expected number of backorders is:

$$\bar{B}_i(R_i, Q_i) = \frac{1}{Q_i}[\beta(R_i) - \beta(R_i + Q_i)] \tag{12}$$

$$\beta(x) = \frac{\sigma^2}{2}\{(z^2+1)[1-\Phi(z)] - z\phi(z)\} \tag{13}$$

$$z = \frac{(x-\theta)}{\sigma} \tag{14}$$

Where θ and σ are the mean and standard deviation of the lead time demand, respectively.

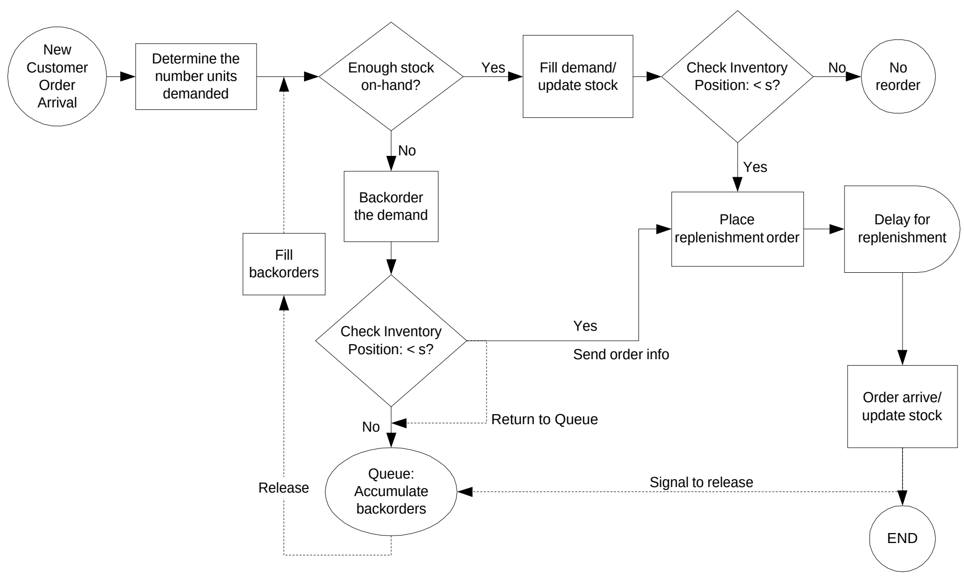

Figure 1. Flowchart of the Single Location Inventory Control Activities, Tee and Rossetti [5]

Tee and Rossetti [5] in an extensive simulation study for multi-echelon inventory systems developed a simulation model for a single item two-echelon inventory system. They studied the robustness of two-echelon (R, Q) analytical inventory models developed by Deuermeyer and Schwarz [4], Svoronos and Zipkin [6], and Axater [3]. The main objective of their study was to examine the analytical models via simulation when the model's basic assumptions are violated. They developed a single item two-echelon identical retailer discrete event inventory simulation model in Arena 5.0 simulation language.

In this study, we rebuilt the simulation model developed by Tee and Rossetti [5] in Arena 9.0 and extended it for the non-identical retailer case. After the simulation model was built in Arena 9.0, step by step debugging was performed. The average inventory investment and the system expected number of backorders are identified as the most important performance measure to be monitored during the simulation experiments. Figure 1 illustrates the logical flow of the model at the retail level. The flow at the warehouse level is essentially the same except that replenishment comes from an external supplier that is assumed to have an infinite supply. In addition, the warehouse satisfies demands sent from the retail level.

4. Experiments and Analysis

A full factorial experiment that investigates the behavior of the inventory system under different parameter conditions usually is required to analyze the effect of each individual factor on the system's performance measures and to measure the interactions between the different factors. An extensive simulation study might be needed before robust conclusions can be derived concerning the effect of the different factors on the behavior of the system under different conditions. For any system, there might be a large number of factors that might affect the behavior of the system under investigation. Only the factors of interest for us will be considered in this simulation experiment. The effect of the retailer's lead-time and demand rate on the average inventory investment and the system's expected number of backorders is considered. Two-levels of the retailer's lead-time (4 and 25 days) and two-levels of the demand rate (55 and 300 unit per year) are considered. The number of the retailers is set at two and the warehouse lead-time is set equal to 4 days. A total of 16 different test cases are developed.

The target average annual order frequencies at retailers 1 and 2 and at the warehouse are 24, 12, and 8, respectively. The target expected number of backorders at retailers 1 and 2 and at the warehouse is 0.5, 0.2, and 0.2, respectively. The expected number of backorders at the retailers are measured in units of demand; while, it is measured at the warehouse in units of the retailer's effective batch size. The item cost is assumed to be $100.00 and does not vary over the retailers and the 16 test cases. The inventory policy parameters are input variables in the simulation model. Instead of considering the policy parameters as factors in the experiments, we used *Algorithm NIR-IHOA* developed by Al-Rifai and Rossetti [2] to set the policy parameters for the 16 test cases.

Initial experiments show that a replication length of 50 years was sufficient to produce uncorrelated batch means. Also, the initial experiments showed that a warm up period of six years was required to mitigate most of the possible initialization bias. However, in order to ensure that the selected warm-up period will capture all the system's initialization bias a 10 year warm up period was selected. Initial on-hand inventory, inventory position, on-order and expected number of backorders at the retailers and the warehouse were set equal to zero.

5. Results and Discussion

The 16 test cases have been simulated where the average inventory investment and expected number of backorders are recorded. The simulation results should represent the actual values of the system performance measures under the simulation assumptions. Since the results obtained using the analytical model are expected to be off from the actual values, we need metrics to compare both results. The error and percentage error between the simulated and analytical model values are calculated. Since the error will show only how much the values via the analytical model are off from the simulated values, we will use the relative error instead of the error in comparing the two models since it is capable of capturing the relative differences between the values. However, the relative error is usually not recommended when the performance measures values are small. A small performance measure value might result in high percentage error, even if the error values are small. For a fixed error, if the performance values increases, the percentage error will decrease.

Table 1, shows that the analytical inventory model overestimated the inventory investment where its percentage error ranged between 10.01% and 49.62% with an average of 25.31%. On the other hand, Table 1 shows that the analytical inventory model underestimated the total system expected number of backorders where its percentage error ranged between -67.49% and -26.58% with an average of -50.69%. Overestimating might be seen as bad as underestimating the system performance measures. Overestimating inventory investment will result in tying more capital that can be used in other investment opportunities. On the other hand, underestimating the inventory investment might result in a bad customer service due to high level of backorders.

Table 1: Analytical vs. simulation results

System	Total Inventory Investment ($)				Expected Number of Backorders (units)			
	Analytical	Simulation	Error	% Error	Analytical	Simulation	Error	% Error
1	824.00	550.72	273.28	49.62%	1.38	2.3423	-0.96	-41.08%
2	1232.05	934.62	297.43	31.82%	1.38	1.8796	-0.50	-26.58%
3	1024.00	736.78	287.22	38.98%	1.38	2.2393	-0.86	-38.37%
4	912.40	634.42	277.98	43.82%	1.38	2.2768	-0.90	-39.39%
5	2349.39	1974.12	375.27	19.01%	2.41	5.6976	-3.29	-57.70%
6	2461.66	2081.24	380.42	18.28%	2.41	5.7718	-3.36	-58.25%
7	2704.90	2316.51	388.39	16.77%	2.41	5.343	-2.93	-54.89%
8	2592.64	2212.74	379.90	17.17%	2.41	5.2785	-2.87	-54.34%
9	2972.35	2472.52	499.83	20.22%	3.43	10.551	-7.12	-67.49%
10	3232.94	2719.60	513.34	18.88%	3.43	9.971	-6.54	-65.60%
11	3320.77	2797.55	523.22	18.70%	3.43	10.0137	-6.58	-65.75%
12	3060.19	2563.58	496.61	19.37%	3.43	10.3969	-6.97	-67.01%
13	4471.96	3634.67	837.29	23.04%	4.45	8.5378	-4.09	-47.88%
14	4733.98	3897.56	836.42	21.46%	4.45	8.4473	-4.00	-47.32%
15	4976.07	4141.90	834.17	20.14%	4.45	8.0742	-3.62	-44.89%
16	4714.05	3865.44	848.61	21.95%	4.45	8.351	-3.90	-46.71%
17	5109.45	4644.55	464.90	10.01%	3.638	5.9125	-2.27	-38.47%
18	7637.10	5842.60	1794.50	30.71%	3.12	7.3867	-4.27	-57.76%
19	9342.89	7085.47	2257.42	31.86%	2.89	7.0592	-4.17	-59.06%
20	14409.41	10727.91	3681.50	34.32%	3.01	7.069	-4.06	-57.42%
Minimum	824.00	550.72	273.28	10.01%	1.38	1.88	-7.12	-67.49%
Maximum	14409.41	10727.91	3681.50	49.62%	4.45	10.55	-0.50	-26.58%
Average	4104.11	3291.73	812.39	25.31%	2.97	6.63	-3.57	-50.69%

The inventory performance measures have a direct effect on setting the inventory policy parameters within an optimization context. Underestimating or overestimating the expected number of backorders affects directly the retailer's effective lead-time. The retailer's effective lead-times are functions of and directly proportional to the expected number of backorders at the warehouse. An increase in the expected number of backorders at the warehouse will result in an increase in the retailer's effective lead-time. Since the retailer's reorder point expressions are functions of the retailer's effective lead-time, as given by Al-Rifai and Rossetti [2], an increase in the retailer's effective lead-time results in an increase in the retailer's reorder point. This indicates that underestimating the expected number of backorders at the warehouse will result in underestimating the retailer's effective lead-time and in turn underestimating the retailer's reordering point.

On the other hand, overestimating the expected number of backorders at the warehouse will result in overestimating the retailer's effective lead-time which in turn will result in overestimating the retailer's reorder point. The retailer's replenishment batch size expressions, as given by Al-Rifai and Rossetti [2], are functions of the expected number of backorders at the warehouse. An increase in the expected number of backorders at the warehouse should result in an increase in the retailer's replenishment batch size and vice-versa. Therefore, overestimating the expected number of backorders at the warehouse results in overestimating the retailer's replenishment batch size and vice versa.

The expected number of backorders at any given location is function of and inversely proportional to its reorder point. Since within the optimization algorithm the search for the reorder point at each location is guided towards its target backorders, underestimating or overestimating the reorder point will directly affect the reorder point obtained using the optimization algorithm. Since the retailer's replenishment batch size expressions, as given by Al-Rifai and

Rossetti [2], are not function of the expected number of backorders at the retailer, overestimating or underestimating the expected number of backorders at the retailer will not affect directly its replenishment batch size. This is also valid at the warehouse. Table 2 summarizes the anticipated effect of overestimating or underestimating the expected number of backorders at any location on the behavior of the optimization algorithm when setting its reorder point.

Table 2: Effect of inaccurately estimating the expected number of backorders

Expected Number of backorders	**Reorder Point**
Overestimated	Underestimated
Underestimated	Overestimated

Since the expected lead-time demand at the warehouse is a function of the effective batch size which is a function of the retailer's batch size, overestimating or underestimating the retailer's batch size will affect the warehouse's performance measures; and in turn, its policy parameters. This is due to the existing dependency between the two echelons. However, the dependency between the two-echelons makes it more complicated to analyze the effect of underestimating or overestimating the performance measures on a particular echelon on the policy parameters at the other echelon. In order to understand such effect, a detailed simulation study that considers analyzing the effect of each performance measure at each echelon on the performance of the optimization algorithm when setting the policy parameters at each echelon may be necessary.

6. Conclusions

This simulation study exposed the quality of the solutions obtained using the performance measures of an analytical model when compared to the actual values. The results showed that over the 16 test cases considered the analytical model overestimated the inventory investment and underestimated the total expected number of backorders. The inventory investment percentage error ranged between 10.01% and 49.62% with an average of 25.31%. On the other hand, the total system expected number of backorders percentage error ranged between -67.49% and -26.58% with an average of -50.69%. Overestimating or underestimating the inventory performance measures impacts directly the behavior of the optimization algorithms. The more accurate inventory performance measures that capture the actual performance measures will results in more accurate policy parameters. Based on our knowledge, we have not seen any analytical inventory models that correctly estimate the inventory performance measures over most of the control variables ranges. Simulation studies similar to this study and the one executed by Tee and Rossetti [5] show that these models are, in most of the cases, off from the simulated values. However, we suggest an extensive research study that considers the most significant factors that might affect the performance of these inventory models. Such a simulation study should result, if possible, in the development of correction factors for such inventory performance measures.

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