An Iterative Heuristic Optimization Model for Multi-Echelon (R, Q) Inventory Systems

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Abstract

Large multi-echelon inventory systems usually consist of hundreds of thousands of stock keep units (SKU). Calculating inventory policies for each product is a computational burden that necessitates the need for more efficient policy setting techniques that reduce computational time and increases managerial convenience. The main objective of our research is to investigate the effect of segmentation (grouping) methodologies on multi-echelon inventory systems. A backorder multi-echelon constrained optimization model is developed that minimizes the expected total system costs. An overview of the research methodology is presented as well as initial results. We also discuss future work intended to examine segmentation policies within this context

Keywords: Optimization, Multi-Echelon, Spare Parts.

1. Introduction

Large multi-echelon, multi-product inventory systems usually consist of hundreds of thousands of stock keep units (SKU). These SKU's can be classified into two main categories: consumables and repairables. Calculating the inventory policy for each SKU is a computational burden that necessitates the need for more efficient policy setting techniques that reduce the computational time, and at the same time improves the ability of item managers to more effectively manage the supply chain. Many industries and large systems depend upon spare (service) parts supply networks to supply them with the required spare parts. Multi-echelon spare parts inventory systems are important to large corporations and to the military to support their field operations. Clustering inventory items into groups based on "operational" and "functional' important attributes is a prominent solution that might reduce computational time and increase management convenience. After clustering the items into groups, a generic group policy might be calculated using a backorder constrained optimization model and applied to every item in the group. In order to set inventory policy we developed and implemented a backorder multi-echelon, multi-product constrained spare parts inventory optimization model that uses reorder point, order quantity inventory policies at the warehouse and retailers. Due to the complexity of the backorder model, we seek to develop expressions that estimate policy parameters at each location at each echelon. In addition, an algorithm must be implemented to arrive at solutions that minimize total inventory investments and satisfy the average order frequency and the average backorder constraints. In developing these expressions and the algorithm we built upon prior work by [5] and [1].

This research is part of a US navy funded project under which we are investigating and developing methodologies for segmenting inventory in large scale multi-echelon inventory systems. In this paper we only develop and present the backorder multi-echelon constrained optimization model that will be used for setting inventory policy parameters while segmenting spare parts. The rest of this paper is organized as follows. In section 2, we review the relevant literature. In section 3, we formulate the optimization model and present an algorithm to approximate inventory policy parameters for each item at each location at each echelon. In section 4, we experiment with the optimization algorithm and provide some conclusions.

2. Literature Review

One of the most important multi-echelon, multi-item inventory models for spare parts management is METRIC. METRIC is the multi-echelon Technique for Recoverable Items Control, developed by Sherbrooke [6] and it is used for setting repairable items inventory control policies using the base stock model. The base stock model is a special case of the reorder point, order quantity (R, Q) inventory policy, where the reorder quantity Q=1 and it is usually used with expensive, slow moving items, and when the holding and backorder costs dominate. In the case of low or medium cost items with medium to high demand rates, the (R, Q) policy may be more appropriate.

Hopp *et al.* [5] considered a two-echelon spare parts stocking and distribution system with an objective function of minimizing total average inventory investment in the entire system subject to constraints on average annual order frequency and total average delay at each facility due to stock out. At the warehouse, they implemented an (R, Q) policy while at each retailer they implemented a base stock model. Since they implemented a base stock model at each retailer, the retailer batch size is equal to one. They assumed that the demand process at each location is Poisson; therefore, the demand process at the warehouse is a superposition of Poisson processes. Hence, the demand rates at the warehouse are known in advance. Since they assumed stochastic retailer leadtimes, service measures at each retailer depend on the delay at the warehouse due to stock outs. The average number of backorders at the warehouse is a function of the inventory policy parameters at the warehouse. Hence, in order to derive expressions that estimate policy parameters at both echelons, they decomposed the system level by level and facility by facility. First, they modeled the warehouse and then they modeled each facility. Decomposition has been widely used in multi-echelon inventory management and queuing systems. Hopp *et al.* [4] considered a single location that implement an (R, Q) policy. Using some approximations and LaGrange multipliers, they arrived at simple exact expressions for inventory policy parameters. Hopp and Spearman [3] presented a multi-product (R, Q) backorder model with an algorithm to estimate inventory policy parameters based on the expressions developed by Hopp *et al.* [4].

Since in this research we implement (R, Q) policies at each location at each echelon, the batch sizes are unknown parameters to be determined by the optimization model. Under a stochastic leadtime assumption and limited supply at the warehouse, the average number of backorders at the warehouse is a function of the warehouse inventory policy parameters. Hence, the effective retailer leadtime is a function of the average number of backorders at the warehouse. In order to set inventory policy parameters at each retailer we need to know a priori the inventory policy parameters at the warehouse. Under (R, Q) policies, the demand rates at the warehouse are functions of each retailer batch size. As we can see, echelons depend on each other. Therefore, to calculate optimal policy parameters, it is necessary to model all the locations and echelons simultaneously. The multi-echelon (R, Q) optimization model presented in the next section is a large scale, non-linear, integer optimization problem [8]. This makes it hard and complicated to model both echelons simultaneously, even using non-linear solvers. The quality of any solution using a non-linear solver depends on the initial parameters values. Hence, we seek to derive expressions that approximate policy parameters at each location at each echelon.

3. Model Formulation and Heuristic Procedure

In order to set inventory policy parameters for each item at each location at each echelon we have formulated an (R, Q) multi-echelon inventory optimization model subject to average order frequency and average number of backorders constraints. For simplicity and due to its wide applicability, we considered a two echelon inventory system, where the lower echelon represents the retailer level with m retailers and the higher echelon represents the warehouse. Retailers are faced by demands that are generated by random failures of spare parts at customer's sites. The spare part failure process is a Poisson process. Since the demand process at each retailer is a Poisson process, the demand process at the warehouse is a superposition of all the retailer's order processes. Specifically, it is a superposition of independent renewal processes each with an Erlang interrenewal processes time with Q_{ri} stages and rate per state λ_{ri} [7]. Before proceeding in developing the model, we list our assumptions as follows:

- We model a two echelon inventory system, where each retailer is replenished by only one warehouse
- Demand process at each retailer occurs according to a Poisson process
- All orders that are not satisfied from shelves are backordered (i.e. lost sales are not considered)
- Limited supply at the warehouse
- Lateral shipments between retailers are not considered
- (R, Q) policy is implemented at each location at each echelon
- Stochastic retailer effective leadtimes
- Fixed warehouse leadtimes. i.e. ample supply at the supplier
- Non-repairable spare parts are considered. i.e. whenever a part is failed, it is discarded
- Zero transportation times for customers at each retailer

The following is a list of the notations that we will use:
F_r : Target order frequency at retailer r

F_w: Target order frequency at the warehouse
B_r: Maximum allowable number of backorders at retailer r
B_w: Maximum allowable number of backorders at the warehouse
m: Number of retailers
N: Number of items
Q_{ri}: Item i replenishment order quantity at retailer r (units)
Q_{wi}: Item i replenishment order quantity at the warehouse (in units of retailer batch size of item i)
R_{ri}: Item i reorder point at retailer r (units)
R_{wi}: Item i reorder point at the warehouse (in units of retailer batch size of item i)
λ_{ri}: Item i demand rate at retailer r (units/year)
λ_{wi}: Item i demand rate at the warehouse (in units of retailer batch size of item i/year)
$\bar{I}_{ri}(R_{ri}, Q_{ri})$: Item i average on-hand inventory at retailer r (units)
$\bar{I}_{wi}(R_{wi}, Q_{wi})$: Item i average on-hand inventory at the warehouse (in units of retailer batch size of item i)
ℓ_{ri}: item i effective leadtime at retailer r (units)
L_{ri}: Item i leadtime (ordering and transportation) at retailer r (years)
L_{wi}: Item i leadtime (ordering and transportation) at the warehouse (years)
C: Total inventory investment at both echelons ($)
c_i: Item i unit cost ($)
r: Retailer index
w: Warehouse index
$\Phi^{-1}(x)$: The inverse of the standard normal cumulative distribution function.
$\eta_r, k_r, \eta_w, \& k_w$: LaGrange multipliers
$t_1, t_2, t_3, \& t_4$: Convergence tolerances.
$\mathrm{V}[D_{wi}]$: Variance of Item i leadtime demand at the warehouse (see Svoronos & Zipkin [7])
$\overline{OF}_{ri}$: Item i average order frequency at retailer r
$\overline{OF}_{wi}$: Item i average order frequency at the warehouse
c & p: Superscripts that represent the current and the precedent steps respectively

We formulate the multi-echelon, multi-item (R, Q) inventory system as minimizing total average inventory investment at both echelons subject to the following average number of backorders and average annual order frequency constraints:

Average annual order frequency at each retailer $\leq \mathrm{F}_r$, r = 1, 2 …m
Average annual number of backorders $\bar{B}_{ri}(R_{ri}, Q_{ri})$ at each retailer $\leq \mathrm{B}_r$, r = 1, 2 …m
Average annual order frequency at the warehouse $\leq \mathrm{F}_w$
Average annual number of backorders $\bar{B}_{wi}(R_{wi}, Q_{wi})$ at the warehouse $\leq \mathrm{B}_w$

We represent the above model mathematically as follows:

$$Minimize \;\; C = \sum_{r=1}^{m}\sum_{i=1}^{N} c_i \bar{I}_{ri}(R_{ri}, Q_{ri}) + \sum_{i=1}^{N} c_i \bar{I}_{wi}(R_{wi}, Q_{wi}) \tag{1}$$

Subject to

$$\frac{1}{N}\sum_{i=1}^{N}\frac{\lambda_{ri}}{Q_{ri}} \leq \mathrm{F}_r, \; r = 1, 2 \ldots m \tag{2}$$

$$\frac{1}{N}\sum_{i=1}^{N}\frac{\lambda_{wi}}{Q_{wi}} \leq \mathrm{F}_w \tag{3}$$

$$\sum_{i=1}^{N}\bar{B}_{ri}(R_{ri}, Q_{ri}) \leq \mathrm{B}_r, \; r = 1, 2 \ldots m \tag{4}$$

$$\sum_{i=1}^{N} \bar{B}_{wi}\left(R_{wi}, Q_{wi}\right) \le \mathrm{B}_w \tag{5}$$

$$R_{ri} \ge -Q_{ri}\,,\ i = 1, 2 \ldots N,\ r = 1, 2 \ldots m \tag{6}$$

$$R_{wi} \ge -Q_{wi}\,,\ i = 1, 2 \ldots N \tag{7}$$

$$Q_{ri} \ge 1\,,\ i = 1, 2 \ldots N,\ r = 1, 2 \ldots m \tag{8}$$

$$Q_{wi} \ge 1\,,\ i = 1, 2 \ldots N \tag{9}$$

$$Q_{ri}, Q_{wi},\ R_{ri},\ \&\ R_{wi}: \text{ Integers, } i = 1, 2 \ldots N,\ r = 1, 2 \ldots m \tag{10}$$

For more details about constraints 6 and 7 refer to [9]. In order to arrive at simple expressions that approximate the policy parameters at each location at each echelon we decompose the system into two levels; the retailer (Model 1) and the warehouse (Model 2) as follows:

Model 1: The retailer

We formulate the optimization problem at the retailer level as minimizing total inventory investment subject to order frequency and backorder constraints as follows.

$$\mathit{Minimize}\ \ C_r = \sum_{r=1}^{m}\sum_{i=1}^{N} c_i \bar{I}_{ri}\left(R_{ri}, Q_{ri}\right) \tag{11}$$

Subject to

$$\frac{1}{N}\sum_{i=1}^{N}\frac{\lambda_{ri}}{Q_{ri}} \le \mathrm{F}_r\,,\ r = 1,\ 2 \ldots m \tag{12}$$

$$\sum_{i=1}^{N} \bar{B}_{ri}\left(R_{ri}, Q_{ri}\right) \le \mathrm{B}_r\,,\ r = 1,\ 2 \ldots m \tag{13}$$

$$R_{ri} \ge -Q_{ri}\,,\ i = 1, 2 \ldots N,\ r = 1, 2 \ldots m \tag{14}$$

$$Q_{ri} \ge 1\,,\ i = 1, 2 \ldots N,\ r = 1, 2 \ldots m \tag{15}$$

$$Q_{ri}\ \&\ R_{ri}: \text{ Integers, } i = 1, 2 \ldots N,\ r = 1, 2 \ldots m \tag{16}$$

Model 2: The warehouse

We formulate the optimization problem at the warehouse as minimizing total inventory investment subject to order frequency and back order constraints as follows.

$$\mathit{Minimize}\ \ C_w = \sum_{i=1}^{N} c_i \bar{I}_{wi}\left(R_{wi}, Q_{wi}\right) \tag{17}$$

Subject to

$$\frac{1}{N}\sum_{i=1}^{N}\frac{\lambda_{wi}}{Q_{wi}} \le \mathrm{F}_w \tag{18}$$

$$\sum_{i=1}^{N} \bar{B}_{wi}\left(R_{wi}, Q_{wi}\right) \le \mathrm{B}_w \tag{19}$$

$$R_{wi} \ge -Q_{wi}\,,\ i = 1, 2 \ldots N \tag{20}$$

$$Q_{wi} \ge 1\,,\ i = 1, 2 \ldots N \tag{21}$$

$$Q_{wi}\ \&\ R_{wi}: \text{ Integers, } i = 1, 2 \ldots N \tag{22}$$

In order to arrive at simple expressions that will be used to estimate inventory policy parameters, we present the following iterative heuristic optimization algorithm:

Multi-Echelon (R, Q) Iterative Heuristic Optimization Algorithm:

Step 1. Set $\ell_{ri} = L_{ri}$, $i = 1, 2 \ldots N$

Step 2. Solve Model 1 as follows:

1. Relax constraints 12 and 13 into the objective function in Model 1
2. Approximate the expected number of backorders at each retailer for each item using a base stock model
3. Solve the resulting LaGrange objective function in Model 1 for each policy parameter as follows:

$$Q_{ri} = \sqrt{\frac{2\eta_r \lambda_{ri}}{Nc_i}}, \; i = 1, 2 \ldots N \tag{23}$$

$$R_{ri} = \sqrt{\lambda_{ri} L_{ri}}\, \Phi^{-1}\left(1 - \frac{c_i}{c_i + k_r}\right) + \lambda_{ri} L_{ri}, \; i = 1, 2 \ldots N \tag{24}$$

4. Search for the LaGrange multipliers that result in policy parameters given by equations 23 & 24 that satisfy all the constraints in Model 1 excluding constraint 16

Step 3. Calculate the expected warehouse leadtime demand (refer to Svoronos and Zipkin [7])

Step 4. Solve Model 2 as follows:

1. Relax constraints 18 and 19 into the objective function in Model 2
2. Approximate the expected number of backorders at the warehouse for each item using a base stock model
3. Solve the resulting LaGrange objective function in Model 2 for each policy parameter as follows:

$$Q_{wi} = \sqrt{\frac{2m\eta_w \lambda_{ri}}{Nc_i Q_{ri}}}, \; i = 1, 2 \ldots N \tag{25}$$

$$R_{wi} = \sqrt{\mathrm{V}[D_{wi}]}\, \Phi^{-1}\left(1 - \frac{c_i}{c_i + k_w}\right) + \frac{\lambda_{ri} m L_{wi}}{Q_{ri}}, \; i = 1, 2 \ldots N \tag{26}$$

4. Search for the LaGrange multipliers that result in policy parameters given by equations 25 & 26 that satisfy all the constraints in Model 2 excluding constraint 22

Step 5. Calculate the expected number of backorders at the warehouse $\overline{B}_{wi}(R_{wi}, Q_{wi})$

Step 6. Calculate the effective retailer leadtime as follows [7]:

$$\ell_{ri} = L_{ri} + \frac{Q_{ri}}{m\lambda_{ri}} \overline{B}_{wi}(R_{wi}, Q_{wi}), \; i = 1, 2 \ldots N \tag{27}$$

Step 7. Resolve Model 1 as follows:

1. Relax constraints 12 and 13 into the objective function in Model 1
2. Approximate the expected number of backorders at each retailer for each item using a base stock model
3. Solve the resulting LaGrange objective function in Model 1 for each policy parameter as follows:

$$Q_{ri} = \sqrt{\frac{\eta_r \lambda_{ri}}{N\left(\frac{c_i}{2} - \frac{\overline{B}_{wi}(R_{wi}, Q_{wi})}{m}\right)}}, \; i = 1, 2 \ldots N \tag{28}$$

$$R_{ri} = \sqrt{\lambda_{ri} \ell_{ri}}\, \Phi^{-1}\left(1 - \frac{c_i}{c_i + k_r}\right) + \lambda_{ri} \ell_{ri}, \; i = 1, 2 \ldots N \tag{29}$$

4. Search for the LaGrange multipliers that result in policy parameters given by equations 28 & 29 that satisfy all the constraints in Model 1 excluding constraint 16

Step 8. Do steps 3-4

Step 9. If $|Q_{ri}^c - Q_{ri}^p| \le t_1$, $|R_{ri}^c - R_{ri}^p| \le t_2$, $|Q_{wi}^c - Q_{wi}^p| \le t_3$, and $|R_{wi}^c - R_{wi}^p| \le t_4$ STOP. Else GO TO step 5 (30)

A good convergence criterion is to check if the absolute difference between the estimated policy parameters in any two consecutive steps is less than or equal to a pre-specified tolerance as shown in Equation 30.

4. Experimentations and Conclusions

In order to test the above model we generate a set of data that represent the multi-echelon inventory system under consideration. We generated the data set shown in Table 1 and implemented the above model in Excel. We used Excel's solver to search for the LaGrange multipliers that result in initial policy parameters that satisfied the average annual order frequency and average number of backorders constraints. For the sake of initial experimentation and testing the above algorithm we set the following target values as follows: $\mathrm{F}_r = 55$, $\mathrm{F}_w = 100$, $\mathrm{B}_r = 15$, and $\mathrm{B}_w = 20$.

We considered only four identical retailers and ten different items. We executed only Steps 1-8 of the above iterative heuristic optimization algorithm. Table 2 shows the results of our initial experimentations. The results shown in Table 2 are feasible since the entire constraints are satisfied. In order to arrive at optimal values (in terms of the tolerance level and the approximations considered), the above Iterative Optimization Algorithm must be coded up and executed until it converges.

Table 1: Cost, demand, and leadtimes data

i	c_i ($)	λ_{ri} (units/year)	L_{ri} (days)	L_{wi} (days)
1	11	1010	1	1
2	2820	7241	23	83
3	971	6169	29	124
4	4045	4802	28	3
5	2927	7299	22	80
6	2402	5198	72	126
7	1755	4117	54	96
8	4480	6646	72	95
9	4115	3287	83	51
10	3734	4913	38	28

Table 2: Experimentation results

					Average Values					
i	*Qri*	*Rri*	*Qwi*	*Rwi*	*Bri(Qri, Rri)*	*OFri*	*Bwi(Qwi, Rwi)*	*OFwi*	*Iri(Rri, Qri) $*	*Iwi(Rwi, Qwi) $*
1	615.7	8.4	5.9	-0.1	0.000	1.6	0.004	1.1	14102.9	22704.9
2	104.1	524.3	2.4	59.7	0.832	69.5	2.401	115.3	664683.5	151477.1
3	163.9	560.8	3.0	48.6	0.161	37.6	1.222	49.8	400090.3	100028.6
4	70.8	398.7	2.0	-0.4	1.368	67.8	1.692	136.4	608531.5	147758.5
5	102.6	503.6	2.4	58.9	0.845	71.1	2.291	118.8	673863.8	155229.8
6	95.6	1096.9	2.3	71.4	1.591	54.4	2.522	94.1	587635.4	117914.2
7	99.5	669.5	2.4	40.5	0.648	41.4	1.877	70.1	453333.3	92278.7
8	79.1	1376.3	2.1	83.0	4.649	84.0	3.391	159.7	770332.7	178793.5
9	58.1	787.4	1.8	28.0	3.323	56.6	2.741	125.6	543897.1	120506.5
10	74.5	547.7	2.0	17.4	1.643	65.9	1.860	129.2	612153.7	143203.6
				Total	**15.0**	**55**	**20.00**	**100**	**$5,328,624.0**	**$1,229,895.5**

Based on our experience, Excel's solver is not powerful enough to solve large scale, non-linear, and mixed-integer models. Since we do not rely on Excel's solver to find optimal or near optimal solutions, the next step of this research is to code-up the above algorithm in Java programming language and implement the multi-product (*R*, *Q*) backorder model developed by [3] to search for the LaGrange multiplier. The above data set is very small; hence we are planning on testing the above algorithm on a large data set that is consistent of thousands of SKU's. The above algorithm is an iterative optimization algorithm in which we assumed fixed retailer leadtimes and approximated the expected number of backorders (in most of the steps) at both echelons using a base stock model. The assumption of fixed retailer leadtime in Step 1 simplified the problem and enabled us to model the retailer without the need for knowing the warehouse policy parameters in advance. Using the policy parameters at the retailer we calculated the expected warehouse leadtime demand which we used to model and set inventory policy parameters at the warehouse. At this point we had initial values for the policy parameters at both echelons. Hence, we could estimate the expected number of backorders at the warehouse which we used to estimate the retailer effective leadtime after relaxing the assumption in Step 1 of the algorithm. This enables us to remodel the retailer as a function of the expected number of backorders at the warehouse. This paper presents the iterative optimization algorithm that estimate policy parameters for each item at each location at each echelon. This algorithm after verified with simulation or optimal solutions enumerated over ranges of policy parameters will be used later for setting inventory policy parameters while segmenting spare parts at large scale multi-echelon inventory systems such as the US Navy.

5. References

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